

# Exploring the Aoki regime

---

**Gernot Akemann and Fabrizio Pucci**

*Fakultät für Physik, Universität Bielefeld, Universitätsstraße 25, D-33615 Bielefeld, Germany*

*E-mail:* [akemann@physik.uni-bielefeld.de](mailto:akemann@physik.uni-bielefeld.de),

[pucci@physik.uni-bielefeld.de](mailto:pucci@physik.uni-bielefeld.de)

**ABSTRACT:** We compute next-to-leading order (NLO) corrections in the  $\epsilon$ -regime of Wilson (WChPT) and Staggered Chiral Perturbation Theory (SChPT). A difference between the two is that in WChPT already at NLO, that is at  $\mathcal{O}(\epsilon^2)$ , new low energy constants (LECs) contribute, whereas in SChPT they only enter at  $\mathcal{O}(\epsilon^4)$ . We first determine the NLO corrections in WChPT for  $SU(2)$ , and for  $U(N_f)$  at fixed index. This implies finite-volume corrections to the phase boundary between the Aoki phase and the Sharpe-Singleton scenario via corrections to the mean field potential. We also compute NLO corrections to the two-point function in the scalar and pseudo-scalar sector in WChPT. Turning to SChPT we determine the NLO corrections to the LECs and their effect on the taste splitting. Here the NLO partition function can be written as the leading order one with renormalized couplings, thus preserving the equivalence to staggered chiral random matrix theory at NLO for any number of flavors  $N_f$ . In WChPT this relation only appears to hold for  $SU(2)$ .

**KEYWORDS:** Wilson and staggered chiral perturbation theory, epsilon-regime, next-to-leading order effects

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Wilson Chiral Perturbation Theory at NLO</b>	<b>3</b>
2.1	Introduction	3
2.2	Partition function for $N_f = 2$ and Aoki Phase	5
2.3	Partition Function for Generic Number of Flavors and Fixed Index	7
2.4	Two-Point Correlation Functions for $N_f = 2$	9
<b>3</b>	<b>Staggered Chiral Perturbation Theory at NLO</b>	<b>13</b>
3.1	Introduction	13
3.2	Staggered Chiral Partition Function at $\mathcal{O}(\epsilon^2)$ for General $N_f$	15
<b>4</b>	<b>Summary and Discussion</b>	<b>18</b>
<b>A</b>	<b>Zero Mode Group Integral Identities</b>	<b>19</b>
A.1	General $SU(N_f)$ Case	19
A.2	The $SU(N_f = 2)$ Case	20
A.3	General $U(N_f)$ Case	20
<b>B</b>	<b>Wilson Chiral Perturbation Theory for General <math>N_f</math> at Fixed Index</b>	<b>21</b>
<b>C</b>	<b>Scalar and Pseudoscalar Currents in WChPT at Fixed Index</b>	<b>24</b>
<b>D</b>	<b>Explicit Computation of Partition Function and Currents for <math>SU(2)</math></b>	<b>27</b>
<b>E</b>	<b>Staggered Chiral Perturbation Theory for General <math>N_f</math></b>	<b>28</b>

---

## 1 Introduction

In the study of the low energy dynamics of QCD it has become of considerable interest to extend chiral perturbation theory by including effects of the lattice spacing  $a$ . Indeed in this way one can control how the ultraviolet cut-off influences every numerical simulations of lattice QCD. For what concerns the Wilson fermion formulation, in a series of works [1, 2, 3, 4, 5] it was shown how the continuum chiral Lagrangian gets modified from discretization effects, so-called Wilson Chiral Perturbation Theory (WChPT). For the staggered version of QCD the analogous extension has been done by Lee and Sharpe [6] for one staggered flavor and then generalized by Aubin and Bernard [7], so-called Staggered Chiral Perturbation Theory (SchPT).

Quite recently there has been a lot of activity in the study of the  $\epsilon$ -regime of WChPT and

SChPT. The discretization effects compete with the quark mass for the explicit breaking of chiral symmetry. Despite the fact that this regime originally introduced in the continuum in [8] is unphysical, which is due to the fact that Compton wave-length is larger than the size of the box, it can be extremely useful in the determination of the low energy effective constants (LECs) and in the study of the Dirac operator spectrum and its dependence on the topology of the gauge fields [9, 10, 11, 12, 13, 14, 15, 16]. Due to the domination of the zero-mode group integral many computations can be performed analytically. First lattice simulations have already confirmed the quenched predictions [17, 18, 19].

Another intriguing characteristic of this regime is that at leading order (LO) in the  $\epsilon$ -expansion it is equivalent to a chiral Random Matrix Theory (ChRMT). More precisely, in the continuum theory a rigorous proof of this equivalence has been given in [20, 21] for all Dirac operator eigenvalue correlation functions. The extension to theories at finite lattice spacing have been derived subsequently: in [10, 11, 12, 13, 14, 15, 16] the authors showed that an analogous equivalence holds between the LO of WChPT and a Wilson Chiral Random Matrix Theory (WChRMT). In [22] this equivalence between the zero-mode sector of SChPT and the corresponding Staggered Chiral Random Matrix Theory (SChRMT) was established.

From the point of view of ChPT when introducing the effects of finite lattice spacing one has to consider the symmetry breaking of the continuum theory down to a subgroup. Consequently more operators will be allowed in the Symanzik effective action, and more terms appear in the chiral Lagrangian. In addition to the chiral condensate  $\Sigma$  and to the pion decay constant  $F$  some new LECs need to be introduced to characterize the strength of these terms. In WChPT in general 3 new LECs are introduced at LO and labeled by  $W_6, W_7, W_8$ , whereas for SChPT 6 new LECs need to be introduced,  $C_1, C_2, C_3, C_4, C_5, C_6$ . The absolute sign of individual LECs and of their combinations has been subject of intense recent discussions in WChPT [11, 23, 24, 25]. It remains to be seen if such arguments carry over to NLO.

In the  $\epsilon$ -regime there are different ways of introducing the relative strength between the mass terms and the lattice spacing. In this paper we will use the so-called Aoki or large cut-off effect (LCE) counting where  $m \sim a^2 \Lambda_{QCD}^3$  or  $a \sim \mathcal{O}(\epsilon^2)$  [26]. Thus the discretization terms are of the same order as the mass terms and compete for the breaking of chiral symmetry already at LO. Considering the Aoki regime means that the LO integral over the zero-modes is modified with respect to the continuum theory and makes the analytic computations more involved. However, one could also consider the Generic Small Mass (GSM) counting [27, 28] where the discretization errors enter only at next-to-next-to-leading order (NNLO) with respect to the continuum Lagrangian, since the counting is  $m \sim a \Lambda_{QCD}^2$  or  $a \sim \mathcal{O}(\epsilon^4)$ . Finally an intermediate regime is known as the GSM\* counting, where  $a \sim \mathcal{O}(\epsilon^3)$  and the discretization errors are at next-to-leading order (NLO) with respect to the continuum [29, 30].

The main question addressed in this paper is to extend WChPT and SChPT to order  $\mathcal{O}(\epsilon^2)$ , for both the partition functions, and for the scalar and pseudoscalar current correlators in the Wilson case. For the spectral density of the Dirac operator a NLO order calculation has already been done, however not in the  $\epsilon$ - but in the  $p$ -regime [31]. In contrast to the

$\epsilon$ -regime in the continuum, sectors of fixed topological charge  $\nu$  are no longer well defined at finite-lattice spacing. They have to be replaced by the index of the Dirac operator, and we refer to [11] for a detailed discussion and references. In the continuum it was found that the NLO partition function could be written as the LO order one with renormalized couplings that include the order  $\mathcal{O}(\epsilon^2)$  effects, both for  $\Sigma$  and for  $F$  when including a iso-spin chemical potential [32, 33, 34]. This implied that the partition function from ChPT and from ChRMT agree up to NLO. Only at NNLO non-universal effects were found in [35]. We will find that for WChPT only for  $SU(2)$  the NLO contributions are absorbed into the two effective couplings relevant in that case, whereas for  $U(N_f)$  at fixed index we have to take extra derivatives of the LO partition function. In contrast for SChPT the NLO contributions can be absorbed into renormalized couplings for any  $N_f$ , due to the  $U(1)$  remnant of the continuum chiral symmetry.

Other information can be extracted from the NLO partition function. For  $SU(2)$  one can see that the finite-volume corrections can drive the system in or out of the Aoki phase, compared to LO. Depending on the coefficients of the mean field potential the theory can stay in the Aoki phase where two pions are massless as a consequence of the breaking of the flavor symmetry, or in the Sharpe-Singleton scenario where a first order transition is present. The finite-volume effects can modify the boundary of these two regions.

For what concerns staggered fermions the effective LECs we compute can lead to the following prediction. Since from the tree level Lagrangian one can see how the taste symmetry is broken, the new renormalized LECs allow to quantify how the finite-volume corrections can modify the taste symmetry violation.

The outline of the paper is as follows. In section 2 we study WChPT at NLO starting with  $SU(2)$  and including the effect on the phase boundary in subsection 2.2, and then turn to  $U(N_f)$  in 2.3. The two-point functions are given in 2.4. In section 3 we repeat our analysis for the staggered version, which includes the effect on the taste splittings. Finally in the last section 4 our discussion and some considerations regarding possible extensions of this work are presented. Several technical details are deferred to the appendices A to E.

## 2 Wilson Chiral Perturbation Theory at NLO

### 2.1 Introduction

In this section we consider the  $\epsilon$ -regime of Wilson Chiral Perturbation theory (WChPT) with  $N_f = 2$  degenerate quarks of mass  $m$ . As already pointed out in the previous section and shown in [11] at LO it is equivalent to a Random Matrix Theory which includes order  $\mathcal{O}(a^2)$  discretization effects (WChRMT). Our aim is to analyze WChPT at the next-to-leading order (NLO) in the  $\epsilon$ -expansion and show that within the Aoki regime the partition function at that order can be rewritten as the LO one with renormalized low energy constants (LECs). In the continuum a similar relation between the LO and NLO partition function holds for every number of flavors as shown e.g. in [32, 33].

Let us start by introducing the two-flavor Wilson chiral Lagrangian that at LO in the Aoki

regime can be written as

$$\mathcal{L}_{\text{LO}} = \frac{F^2}{4} \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] - \frac{\Sigma}{2} \text{Tr} \left[ M^\dagger U + U^\dagger M \right] + a^2 c_2 \left( \text{Tr} \left[ U + U^\dagger \right] \right)^2. \quad (2.1)$$

In addition to the continuum Gasser-Leutwyler terms [36, 37] there is an additional order  $\mathcal{O}(a^2)$  contribution and thus a new low energy effective constant  $c_2$  (note that our  $c_2 = W_6 + W_8/2$  is a short hand notation for the standard terminology which is  $c_2 F^2/16$ , and likewise for the other NLO LECs). Here as usual  $F$  is the pion decay constant,  $\Sigma$  is the chiral condensate and  $M$  is the mass matrix that, for two degenerate quarks with mass  $m$ , reduces to  $M = m \mathbb{I}_{2 \times 2}$ . In all the following we will only consider degenerate masses. We use the standard parameterization for the Goldstone boson

$$U(x) = U_0 \exp \left[ i \frac{\sqrt{2}}{F} \xi(x) \right], \quad (2.2)$$

where  $U_0$  is the two by two unitary matrix describing the zero-modes nonperturbatively, and the Hermitian fields  $\xi(x) = \xi(x)^\dagger = \sigma_b \xi_b$ , that belong to the Lie algebra  $su(2)$ , parameterize the propagating modes. In order to derive the Lagrangian (2.1) in the so-called Aoki regime one has to use the power counting [26]

$$V \sim \epsilon^{-4}, \quad m \sim \epsilon^4, \quad \partial \sim \epsilon, \quad \xi(x) \sim \epsilon, \quad a \sim \epsilon^2. \quad (2.3)$$

Other counting schemes can be considered if we want to study the GSM\* or GSM regime [27, 28] that are in fact defined considering the cut-off effects respectively as NLO and NNLO contributions with respect to the continuum terms.

If we want to go further in the Aoki regime and compute the NLO partition function we have to consider in addition to (2.1) the NLO chiral Lagrangian.

Leading Order $\mathcal{O}(\epsilon^0)$	$m, p^2, a^2$
Next-to-Leading Order $\mathcal{O}(\epsilon^2)$	$am, ap^2, a^3$
Next-to-Next-to-Leading Order $\mathcal{O}(\epsilon^4)$	$m^2, mp^2, p^4, a^2m, a^2p^2, a^4$

**Table 1.** Contributions to the Wilson Chiral Lagrangian in the Aoki regime.

As we can see from the table 1. that schematically gives us the counting of all the terms that contribute up to  $\mathcal{O}(\epsilon^4)$  to the chiral Lagrangian, at order  $\mathcal{O}(\epsilon^2)$  the possible terms that enter are  $\mathcal{O}(ap^2)$ ,  $\mathcal{O}(am)$  and  $\mathcal{O}(a^3)$ . Following [2, 24, 38, 39] they can be written as

$$\begin{aligned} \mathcal{L}_{\text{NLO}} = & a c_0 \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] \text{Tr} \left[ U + U^\dagger \right] + am c_3 \left( \text{Tr} \left[ U + U^\dagger \right] \right)^2 \\ & + a^3 d_1 \text{Tr} \left[ U + U^\dagger \right] + a^3 d_2 \left( \text{Tr} \left[ U + U^\dagger \right] \right)^3 \end{aligned} \quad (2.4)$$

where for  $SU(2)$  4 new and undetermined LECs, namely  $c_0, c_3, d_1$  and  $d_2$ , need to be introduced. Here we have explicitly used some special properties of  $SU(2)$ , see e.g. appendix A, compared to the general  $N_f$  case. Now using the power counting (2.3) we expand the action

$$S = \int d^4x \left( \mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} \right) \quad (2.5)$$

up to  $\mathcal{O}(\epsilon^2)$ , where we obtain

$$\begin{aligned} S^{(0)} &= \frac{1}{2} \int d^4x \operatorname{Tr} [\partial_\mu \xi(x) \partial_\mu \xi(x)] - \frac{1}{2} m V \Sigma \operatorname{Tr} [U_0 + U_0^\dagger] + a^2 V c_2 \left( \operatorname{Tr} [U_0 + U_0^\dagger] \right)^2 \\ &\equiv S_{\partial^2}^{(0)} + S_{U_0}^{(0)} \end{aligned} \quad (2.6)$$

for the  $\mathcal{O}(\epsilon^0)$  contribution. Here we have defined the LO part of propagating and zero-modes separately. For the  $\mathcal{O}(\epsilon^2)$  terms we get

$$\begin{aligned} S^{(2)} &= \frac{1}{12 F^2} \int d^4x \operatorname{Tr} [\partial_\mu \xi(x), \xi(x)] [\partial_\mu \xi(x), \xi(x)] + \frac{m \Sigma}{2 F^2} \int d^4x \operatorname{Tr} \left[ (U_0 + U_0^\dagger) \xi(x)^2 \right] \\ &\quad - 2a^2 \frac{c_2}{F^2} \int d^4x \left( \operatorname{Tr} \left[ (U_0 - U_0^\dagger) \xi(x) \right] \right)^2 \\ &\quad - 2a^2 \frac{c_2}{F^2} \operatorname{Tr} [U_0 + U_0^\dagger] \int d^4x \operatorname{Tr} \left[ (U_0 + U_0^\dagger) \xi(x)^2 \right] \\ &\quad + \frac{2a c_0}{F^2} \operatorname{Tr} [U_0 + U_0^\dagger] \int d^4x \operatorname{Tr} [\partial_\mu \xi(x) \partial_\mu \xi(x)] + a m c_3 V \left( \operatorname{Tr} [U_0 + U_0^\dagger] \right)^2 \\ &\quad + a^3 d_1 V \operatorname{Tr} [U_0 + U_0^\dagger] + a^3 d_2 V \left( \operatorname{Tr} [U_0 + U_0^\dagger] \right)^3, \end{aligned} \quad (2.7)$$

while the  $\mathcal{O}(\epsilon)$  term vanishes due to  $\int d^4x \xi(x) = 0$ .

## 2.2 Partition function for $N_f = 2$ and Aoki Phase

The next step is the calculation of the partition function up to  $\mathcal{O}(\epsilon^2)$ . The general form of the partition function can be rearranged by separating the integration over the zero-modes from the integration over the Gaussian fluctuations as

$$\mathcal{Z} = \int_{SU(2)} [d_H U(x)] e^{-S} = \int_{SU(2)} d_H U_0 e^{-S_{U_0}^{(0)}} \mathcal{Z}_\xi(U_0), \quad (2.8)$$

with

$$\mathcal{Z}_\xi(U_0) = \int [d\xi(x)] \left( 1 - \frac{2}{3F^2 V} \int d^4x \operatorname{Tr} [\xi(x)^2] \right) e^{S_{U_0}^{(0)} - S}, \quad (2.9)$$

containing the Jacobian  $J(\xi(x))$  up to order  $\mathcal{O}(\epsilon^3)$  [8] from the parameterization (2.2), and the chiral action  $S$  to an unspecified order. Here the invariant Haar measure

$$[d_H U(x)] = d_H U_0 [d\xi(x)] \left( 1 - \frac{N_f}{3F^2 V} \int d^4x \operatorname{Tr} [\xi(x)^2] \right) \quad (2.10)$$

has been divided as the invariant measure over the zero-modes  $U_0$  times the flat measure over the fluctuation  $\xi$ . At this point one can expand the function  $\mathcal{Z}_\xi(U_0)$  up to  $\mathcal{O}(\epsilon^2)$  and then perform all the Gaussian integrals using the expression

$$\int [d\xi(x)] \exp[-S_{\partial^2}^{(0)}] \xi(x)_{ij} \xi(y)_{kl} = \left( \delta_{il} \delta_{jk} - \frac{1}{N_f} \delta_{ij} \delta_{kl} \right) \Delta(x - y) \quad (2.11)$$

in terms of the propagator. We easily find that

$$\begin{aligned} \mathcal{Z}_\xi(U_0) &= \mathcal{N} \left\{ 1 + \left( -\frac{3mV\Sigma}{4F^2} \Delta(0) - a^3 d_1 V \right) \operatorname{Tr} [U_0 + U_0^\dagger] \right. \\ &\quad \left. + \left( \frac{4a^2 c_2 V}{F^2} \Delta(0) - a m c_3 V \right) \left( \operatorname{Tr} [U_0 + U_0^\dagger] \right)^2 - a^3 d_2 V \left( \operatorname{Tr} [U_0 + U_0^\dagger] \right)^3 \right\}. \end{aligned} \quad (2.12)$$

Here  $\mathcal{N}$  is an overall normalization factor that contains all constants that are  $U_0$  independent and that drop out in expectations values. In particular this includes the contribution from the Jacobian. The propagator  $\Delta(0)$  is finite in dimensional regularization and is given by  $\Delta(0) = -\beta_1/V^{1/2}$ , with  $\beta_1$  a numerical coefficient that encodes the geometrical data of the box considered. At this point we note that all terms in (2.12) can be reabsorbed easily in the LO chiral Lagrangian by re-exponentiating the corrections, with the only the exception of the last term. In order to solve this problem one can write this contribution as a sum of single and double trace terms using the relation (A.8) obtained in the appendix A.2 through some group integral identities. Finally the partition function can be written as

$$\begin{aligned} \mathcal{Z}_{\text{NLO}} &= \mathcal{N}' \int_{SU(2)} d_H U_0 \exp \left[ \frac{m\Sigma^{\text{eff}}V}{2} \text{Tr} [U_0 + U_0^\dagger] - a^2 c_2^{\text{eff}} V \left( \text{Tr} [U_0 + U_0^\dagger] \right)^2 \right] \\ &= \frac{\mathcal{N}'}{\mathcal{N}} \mathcal{Z}_{\text{LO}}(\Sigma^{\text{eff}}, c_2^{\text{eff}}), \end{aligned} \quad (2.13)$$

with the effective renormalized LECs given by

$$\Sigma^{\text{eff}} = \Sigma \left( 1 - \frac{3}{2F^2} \Delta(0) - \frac{\hat{a}}{\hat{m}\sqrt{V}} \left( 2\hat{a}^2 d_1 + 32\hat{a}^2 d_2 - 3\frac{d_2}{c_2} \right) \right), \quad (2.14)$$

and

$$c_2^{\text{eff}} = c_2 \left( 1 - \frac{4}{F^2} \Delta(0) \right) + \frac{\hat{m}}{\hat{a}} \left( \frac{c_3}{\Sigma} + \frac{d_2}{4c_2} \right) \frac{1}{\sqrt{V}}. \quad (2.15)$$

Here we have defined

$$\hat{m} \equiv m\Sigma V \quad \text{and} \quad \hat{a}^2 \equiv a^2 V, \quad (2.16)$$

which are of order  $\mathcal{O}(1)$ . Differently from the continuum limit chiral perturbation theory, in which the NLO renormalized LECs can be rewritten only as functions of the number of flavors and the geometry of the system, here their expressions involve also some NLO LECs. In principle this allows us to extract them from lattice computations through a finite-size scaling analysis. Performing the simulations at two different lattice volumes  $V_1$  and  $V_2$ , with geometries  $\beta_1$  and  $\beta_2$ , WChPT predicts a scaling of the LECs as

$$\frac{\Sigma^{\text{eff}}(V_1)}{\Sigma^{\text{eff}}(V_2)} = 1 + \frac{3}{2F^2} \frac{(\beta_1\sqrt{V_2} - \beta_2\sqrt{V_1})}{\sqrt{V_1 V_2}} + \left( \frac{3ad_2}{mc_2\Sigma} \right) \left( \frac{1}{V_1} - \frac{1}{V_2} \right) \quad (2.17)$$

$$\frac{c_2^{\text{eff}}(V_1)}{c_2^{\text{eff}}(V_2)} = 1 + \frac{4}{F^2} \frac{(\beta_1\sqrt{V_2} - \beta_2\sqrt{V_1})}{\sqrt{V_1 V_2}}. \quad (2.18)$$

From the NLO partition function we can also extract information about the Aoki phase. The possible existence of such a phase in which flavor symmetry can be broken (with no analogon in the continuum theory) has been an outstanding problem for a long time. Quite recently in [25] the authors showed that while in the unquenched theory both scenarios (the Aoki and Sharpe-Singleton scenario) can be realized, the quenched theory at sufficiently small quark mass is always in the Aoki phase (see [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50] for lattice data).

Morally speaking in the infinite-volume case flavor symmetry breaking can occur due to the

fact that  $\langle U_0 \rangle \neq 0$ . Let us repeat here the analysis of [1] and apply it to our NLO results. In order to determine the value of the minimum of the potential energy we parameterize  $U_0 = A + iB_j \cdot \sigma_j$ , with  $\sigma_j$  the Pauli matrices. This makes the action in eq. (2.13) only depend on  $A$ . If we assume that the sign of  $c_2$  is positive the potential will be a parabola, and the minimum is given by the parameter usually called

$$\hat{\varepsilon} = \frac{m\Sigma}{16a^2c_2} = \frac{\hat{m}}{16\hat{a}^2c_2}. \quad (2.19)$$

If this parameter lies outside the range  $-1$  to  $1$  then it is simple to see that the vector symmetry can not be spontaneously broken, and the minimum is taken by  $A = 1$ . However, if the minimum satisfies  $|\hat{\varepsilon}| < 1$  the vacuum is determined by  $A^* = \hat{\varepsilon}$ . As a consequence  $B_j^* \neq 0$  and flavor symmetry is spontaneously broken to  $U(1)$ . This tells us that the region  $-1 < \hat{\varepsilon} < 1$  has the properties of the Aoki phase. We denote the value at which the transition takes place by  $c_2^* = \frac{\hat{m}}{16\hat{a}^2}$ .

One can repeat the same analysis using our NLO partition function to analyze the role of the finite-volume corrections to this picture. The parameter  $\hat{\varepsilon}$  is obviously modified at finite  $V$  and more precisely it is given by

$$\begin{aligned} \hat{\varepsilon}_V &= \frac{\hat{m}\Sigma^{\text{eff}}}{16\hat{a}^2\Sigma c_2^{\text{eff}}} \\ &= \frac{\hat{m}}{16\hat{a}^2c_2} \left( 1 + \frac{4}{2F^2}\Delta(0) - \frac{\hat{a}}{\hat{m}\sqrt{V}} \left( 2\hat{a}^2d_1 + 32\hat{a}^2d_2 - 3\frac{d_2}{c_2} \right) - \frac{\hat{m}}{\hat{a}} \left( \frac{c_3}{\Sigma} + \frac{d_2}{4c_2} \right) \frac{1}{\sqrt{V}} \right). \end{aligned} \quad (2.20)$$

The finite-volume corrections can thus change the phase of the system since they modify the range of the minimum of the parabola determining the potential energy of the system.

### 2.3 Partition Function for Generic Number of Flavors and Fixed Index

In this section we will consider Wilson Chiral Perturbation Theory with  $N_f$  flavors. The situation becomes more complicated since additional terms are allowed in the Lagrangian. Indeed the Wilson chiral Lagrangian for a generic number of degenerate quark flavors with mass  $m$  reads at LO

$$\begin{aligned} \mathcal{L}_{\text{LO}} &= \frac{F^2}{4} \text{Tr} \left( \partial_\mu U \partial_\mu U^\dagger \right) - \frac{m\Sigma}{2} \text{Tr} \left( U + U^\dagger \right) - \frac{z\Sigma}{2} \text{Tr} \left( U - U^\dagger \right) \\ &\quad + a^2 W_8 \text{Tr} \left( U^2 + U^{\dagger 2} \right) + a^2 W_6 \text{Tr} \left( U + U^\dagger \right)^2 + a^2 W_7 \text{Tr} \left( U - U^\dagger \right)^2, \end{aligned} \quad (2.21)$$



where we have introduced a source  $z$  for the axial quark mass, and for the NLO [24]<sup>1</sup> it reads

$$\begin{aligned}
\mathcal{L}_{\text{NLO}} = & a w_4 \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] \text{Tr} \left[ U + U^\dagger \right] + a w_5 \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \left( U + U^\dagger \right) \right] \\
& - a m w_6 \left( \text{Tr} \left[ U + U^\dagger \right] \right)^2 - a m w_7 \left( \text{Tr} \left[ U - U^\dagger \right] \right)^2 - a m w_8 \text{Tr} \left[ U^2 + U^{\dagger 2} \right] \\
& + a^3 x_1 \left( \text{Tr} \left[ U + U^\dagger \right] \right)^3 + a^3 x_2 \left( \text{Tr} \left[ U - U^\dagger \right] \right)^2 \text{Tr} \left[ U + U^\dagger \right] \\
& + a^3 x_3 \text{Tr} \left[ U^2 + U^{\dagger 2} \right] \text{Tr} \left[ U + U^\dagger \right] + a^3 x_4 \text{Tr} \left[ U^2 - U^{\dagger 2} \right] \text{Tr} \left[ U - U^\dagger \right] \\
& + a^3 x_5 \text{Tr} \left[ U^3 + U^{\dagger 3} \right] + a^3 x_6 \text{Tr} \left[ U + U^\dagger \right]. \tag{2.22}
\end{aligned}$$

At this point one can proceed following the same steps used in the  $N_f=2$  analysis, namely expand the action up  $\mathcal{O}(\epsilon^2)$  and integrate out the pion fluctuation. Just for stylistic reasons we report our detailed computation in appendix B. Here we present only the final result and make some clarifications.

As we can see from both the LO and NLO Lagrangians (2.21) and (2.22) a lot of terms appear compared to the usual  $N_f = 2$  case. At LO there are three independent LECs, *i.e.*  $W_6, W_7$  and  $W_8$ , and we recall again that in the simple case of two flavors they combine just in one coefficient called  $c_2 = W_6 + W_8/2$  while  $W_7$  doesn't enter in the computation. At NLO we have eleven new coefficients that are divided as follows. We have order  $\mathcal{O}(p^2 a)$ ,  $w_4$  and  $w_5$  written in the first line of eq. (2.22). Again one can show that for the case of two flavors these two terms combine in one contribution which coefficient that we called  $c_0 = w_4 + w_5/2$ . Then there are three terms of order  $\mathcal{O}(ma)$  whose coefficients are  $w_6, w_7$  and  $w_8$ , and the expressions are listed in the second line of the same equation. Again for two flavors  $w_7$  doesn't contribute and the other terms form a combination that we called  $c_3 = w_6 + w_8/2$  previously. And finally there are the more tedious contributions of order  $\mathcal{O}(a^3)$  with six different terms and the relative six coefficient  $x_i$  with  $i$  running from one to six. In the case of two flavors, only two are independent and in particular the non-trivial combinations  $d_1 = x_1 + x_3/2 + x_5/4$  and  $d_2 = x_6 - 4x_3 - 3x_5$  enter in the game, whereas  $x_2$  and  $x_4$  don't contribute.

Since we will work at fixed index  $\nu$  we define the projection to the following Fourier components:

$$\mathcal{Z}^\nu \equiv \int_{U(N_f)} [d_H U(x)] \det[U(x)^\nu] e^{-S}. \tag{2.23}$$

Taking the Fourier sum over all components will lead back to the  $SU(N_f)$  integral. Note that after fixing the index the  $N_f = 2$  case has as many terms in the Lagrangian as for general  $N_f$ , as there are no identities left to simplify it. At LO this partition function can

---

<sup>1</sup>Note that compared to [24] we have absorbed a factor of  $2\Sigma/F^2$  into our  $w_j$ .

be evaluated as

$$\begin{aligned} \mathcal{Z}_{\text{LO}}^\nu = \mathcal{N} \int_{U(N_f)} d_H U_0 \det[U_0^\nu] \exp \left[ \frac{m\Sigma V}{2} \text{Tr}[U_0 + U_0^\dagger] + \frac{z\Sigma V}{2} \text{Tr}[U_0 - U_0^\dagger] \right. \\ \left. - a^2 V W_8 \text{Tr}[U_0^2 + U_0^{\dagger 2}] - a^2 V W_6 \left( \text{Tr}[U_0 + U_0^\dagger] \right)^2 - a^2 V W_7 \left( \text{Tr}[U_0 - U_0^\dagger] \right)^2 \right], \end{aligned} \quad (2.24)$$

with expectation values defined by

$$\langle F(U_0) \rangle_{\text{LO}} = \frac{1}{\mathcal{Z}_{\text{LO}}^\nu} \int_{U(N_f)} d_H U_0 F(U_0) \det[U_0^\nu] e^{-S_{U_0}^{(0)}}, \quad (2.25)$$

using the corresponding action  $S_{U_0}^{(0)}$  from eq. (2.24). The group integral eq. (2.24) is known explicitly and given in appendix C.

Since we have to take expectation values with respect to a  $U(N_f)$  integral instead of  $SU(N_f)$  we have derived new identities between the expectation values of the various terms in appendix A.3.

After expanding the action up to  $\mathcal{O}(\epsilon^2)$  and integrating out the fluctuations it becomes non-trivial to reabsorb all the terms in such a way that the NLO Lagrangian can be expressed in terms of the LO one. Indeed we have to make use of three relations listed in the appendix A.3 to find that

$$\begin{aligned} \mathcal{Z}_{\text{NLO}}^\nu = \frac{\mathcal{N}''}{\mathcal{N}} \left( \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}^{\text{eff}}, \hat{z}^{\text{eff}}, \hat{a}_6^{\text{eff}}, \hat{a}_7^{\text{eff}}, \hat{a}_8^{\text{eff}} \right) + X_1^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_6^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \right. \\ \left. + X_2^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_7^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) + X_3^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_6^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \right. \\ \left. - 4X_5^{\text{eff}} \frac{\partial^2}{\partial \hat{z} \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \right), \end{aligned} \quad (2.26)$$

where the explicit expressions for the renormalized constants  $\hat{m}^{\text{eff}}, \hat{z}^{\text{eff}}, \hat{a}_6^{\text{eff}}, \hat{a}_7^{\text{eff}}, \hat{a}_8^{\text{eff}}$ , as well as for  $X_1^{\text{eff}}, X_2^{\text{eff}}, X_3^{\text{eff}}, X_5^{\text{eff}}$  are derived in the appendix B.

When we set the source  $\hat{z} = 0$  the last effective coupling vanishes,  $X_5^{\text{eff}}|_{z=0} = 0$ . Also note if we were to set all the extra LECs to zero that contribute to the chiral Lagrangian at NLO (as it happens for SchPT in the next section), that is  $x_{1,\dots,6} = 0 = w_{6,7,8}$ , then all  $X_{1,2,3,5}^{\text{eff}} = 0$  would equally vanish, and we could again write the NLO partition function as a LO one with the couplings renormalized through the one-loop corrections. Also the masses  $m$  and  $z$  would then be renormalized with the same effective LEC  $\Sigma^{\text{eff}}$ . This is the situation we find below in SchPT, and it is also true for the NLO finite-volume corrections in the continuum.

## 2.4 Two-Point Correlation Functions for $N_f = 2$

In this subsection we will calculate the two-point correlators of the scalar and pseudoscalar current densities, in analogy to the continuum results in [51, 52, 53]. They are defined

respectively by

$$S_0(x) = \bar{\psi}(x)\psi(x) \quad , S_b(x) = \bar{\psi}(x)t_b\psi(x) \quad (2.27)$$

$$P_0(x) = i\bar{\psi}(x)\gamma_5\psi(x) \quad , P_b(x) = i\bar{\psi}(x)t_b\gamma_5\psi(x) \quad (2.28)$$

at the first non-trivial order in the  $\epsilon$ -expansion. In the previous expression  $t_b$  are proportional to the Pauli matrices  $t_b = \frac{1}{2}\sigma_b$  for  $b = 1, 2, 3$ . Following the standard procedure, in order to calculate these quantities one has to introduce the Hermitian sources  $s$  and  $p$  in the partition function through the replacement  $M \rightarrow M + s_0(x) + s_b(x)t_b + ip_0(x) + ip_b(x)t_b$ , and take the following functional derivatives:

$$\langle S_b(x)S_c(0) \rangle = \frac{1}{\mathcal{Z}} \frac{\delta^2}{\delta s_b(x)\delta s_c(0)} \mathcal{Z}[s, p] \big|_{s=p=0} , \quad (2.29)$$

$$\langle P_b(x)P_c(0) \rangle = \frac{1}{\mathcal{Z}} \frac{\delta^2}{\delta p_b(x)\delta p_c(0)} \mathcal{Z}[s, p] \big|_{s=p=0} . \quad (2.30)$$

Before starting we recall that the isovector scalar ( $s_{b=1,2,3}(x)$ ) and the isoscalar pseudoscalar ( $p_0(x)$ ) densities, as a property of the  $SU(2)$  theory, are vanishing at LO and also NLO and thus we will not consider these quantities.

The additional sources lead to the following modification of the LO Lagrangian eq. (2.1)  $\mathcal{L}_{\text{LO}} \rightarrow \mathcal{L}_{\text{LO}} + \delta\mathcal{L}_{\text{LO}}$  with

$$\delta\mathcal{L}_{\text{LO}} = -s_0(x)\frac{\Sigma}{2}\text{Tr}[U + U^\dagger] + ip_b(x)\frac{\Sigma}{2}\text{Tr}[t_b(U - U^\dagger)] , \quad (2.31)$$

as well as to the corresponding modification of the NLO Lagrangian eq. (2.4)

$$\delta\mathcal{L}_{\text{NLO}} = s_0(x)ac_3 \left( \text{Tr}[U + U^\dagger] \right)^2 - ip_b(x)ac_3 \text{Tr}[t_b(U - U^\dagger)] \text{Tr}[U + U^\dagger] . \quad (2.32)$$

In addition to the continuum the two-point functions are only known up to the renormalization constants  $W_{S,P}^b$ , that are of the form  $(1 + aW_S^b)\langle S_b(x)S_c(0) \rangle$ , and likewise for the pseudoscalars (see [54] for details about the renormalization procedure).

Now we have to calculate the correlators at NLO in the  $\epsilon$ -expansion. First, we expand the observables up to  $\mathcal{O}(\epsilon^2)$  using the NLO action, in the second step we perform the Gaussian integration over the fluctuations. Here we report only the results for  $SU(2)$  while we show explicitly the full calculations for  $U(N_f)$  at fixed index in the appendix C.

The advantage of the fixed index averages is that we could use the compact integral representations for the LO partition function, eq. (C.4) or eq. (C.8) for  $N_f = 2$ , to obtain the NLO expressions from eq. (2.26). The logarithmic derivatives with respect to the corresponding couplings then generate all group averages given explicitly in appendix 2.4 in eqs. (C.9), (C.10) from these LO and NLO partition functions.

For practical purposes however the disadvantage of fixed index at NLO is the large number of LECs to enter the expressions, that is for  $N_f = 2$  the 3 LECs  $W_{6,7,8}$  from LO plus an additional 6 combinations from NLO in eq. (2.26). For this reason we have not attempted

to plot the NLO two-point functions calculated at fixed index.

For the two-point function of the scalar current density we obtain

$$\begin{aligned} \langle S_0(x)S_0(0) \rangle = & \frac{(\Sigma^{\text{eff}})^2}{4} \left\langle \left( \text{Tr}[U_0 + U_0^\dagger] \right)^2 \right\rangle_{\text{NLO}} - \frac{\Sigma^2}{2F^2} \left\langle \text{Tr} \left[ U_0^2 + U_0^{\dagger 2} \right] - 4 \right\rangle_{\text{LO}} \Delta(x) \\ & - \frac{\hat{a}\Sigma c_3}{\sqrt{V}} \left\langle \left( \text{Tr}[U_0 + U_0^\dagger] \right)^3 \right\rangle_{\text{LO}}, \end{aligned} \quad (2.33)$$

where the averages are now over constant matrices  $U_0 \in SU(2)$  with the NLO or LO partition function eq. (2.13), respectively. Apart from the averaging partition function the expression in the first line completely agrees with the continuum expression, see e.g. [33], after setting  $N_f = 2$  with  $\text{Tr} [U_0 - U_0^\dagger] = 0$  there.

For the pseudoscalar sector we have

$$\begin{aligned} \langle P_b(x)P_b(0) \rangle = & -\frac{(\Sigma^{\text{eff}})^2}{8} \left\langle \text{Tr} \left[ U_0^2 + U_0^{\dagger 2} \right] - 4 \right\rangle_{\text{NLO}} \\ & + \frac{\Sigma^2}{4F^2} \left\langle \frac{3}{4} \left( \text{Tr} [U_0 + U_0^\dagger] \right)^2 - \text{Tr} [U_0^2 + U_0^{\dagger 2}] + 4 \right\rangle_{\text{LO}} \Delta(x) \\ & + \frac{\hat{a}c_3\Sigma}{2\sqrt{V}} \left\langle \left( \text{Tr} [U_0^2 + U_0^{\dagger 2}] - 4 \right) \text{Tr} [U_0 + U_0^\dagger] \right\rangle_{\text{LO}}, \end{aligned} \quad (2.34)$$

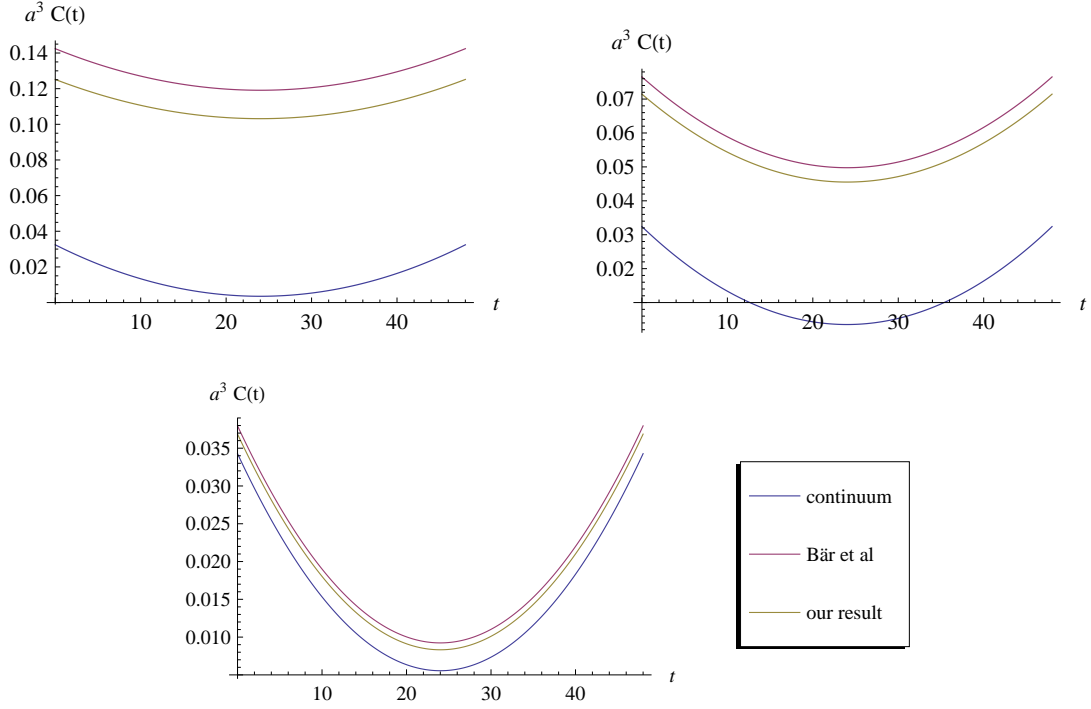
where we have used the  $su(n_f)$  completeness relation eq. (A.2) after summing over  $b = 1, 2, 3$ . Once again the first two lines agree with the continuum expression for  $N_f = 2$ , apart from the different average.

Let's start by plotting the zero-momentum correlator for different values of the mass  $m$  and at fixed values of  $c_2$  and lattice spacing  $a$ . More in detail we consider a hypercubic symmetric lattice with  $N_L = N_T = 48$  ( $\beta_1 = 0.140461$ ) and use  $F = 90$  MeV and  $\Sigma = 250$  MeV. In table 2. we list all parameters and numerical values of the LECs used in the plots (for details on the explicit integrals used see appendix D).

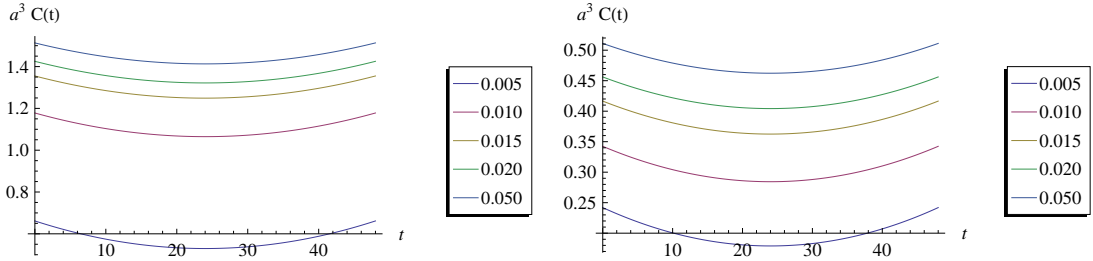
	$a(\text{fm})$	$m(\text{MeV})$	$c_2(\text{GeV}^4)$	$d_1(\text{GeV}^4)$	$d_2(\text{GeV}^4)$	$c_3(\text{GeV}^4)$
Fig. 1.	0.08	7, 2, 0.5	0.01	$10^{-5}$	$10^{-5}$	$10^{-5}$
Fig. 2.	0.12, 0.10	1	0.005 ... 0.050	$10^{-5}$	$10^{-5}$	$10^{-5}$
Fig. 3.	0.10	1	-0.1 ... 0.1	$10^{-2}$	0	0

**Table 2.** Numerical values of the lattice spacing, masses and LECs used in the evaluation of the two-point function from appendix D.

In figure 1. we plot the integrated two-point correlator, eqs. (2.34) and (2.35), comparing our results with the continuum limit [51, 52, 53], and with the results obtained by Bär, Necco and Schaefer [29] (see also [30]) that are valid in the GSM\* regime, where finite lattice spacing effects only enter at NLO, compared to LO in our LCE counting. We only display the pseudoscalar-correlator, at three different masses with otherwise fixed parameters: for the first two values chosen,  $m=0.5$  and  $m=2$  MeV the system is in the Aoki regime (top



**Figure 1.** Pseudoscalar correlators at fixed  $a=0.08$  and different masses  $m=0.5$  (top-left),  $m = 2$  (top-right) and  $m = 7$  MeV (bottom). The different colors distinguish the continuum result, the results of [29] in the GSM\* regime and our result.



**Figure 2.** Pseudoscalar correlators at fixed  $a=0.12$  (left),  $a = 0.10$  (right) and fixed mass  $m = 1$  MeV, for different values of the LEC  $c_2$  encoded by different colors.

two plots in fig. 1.) while for the last one with  $m=7$  MeV it is in the GSM\* regime instead (bottom fig. 1). Our calculation agrees quite well with [29, 30] when the GSM\* counting is valid, while it disagrees when one enters the Aoki regime. Indeed the GSM\* expansion is then no more reliable since lattice spacing effects give LO contributions and cannot be considered as perturbations. Similar plots could be obtained for the scalar two-point function.

In figure 2. we plot the same pseudoscalar correlator for different values of  $c_2$  ranging from 0.005 to 0.05 GeV<sup>4</sup>, at fixed  $m$  and  $a$ . In the left figure we use a lattice spacing of  $a = 0.12$  fm while in the right plot we have  $a = 0.08$  fm. As one could expect for the bigger lattice spacing small changes of  $c_2$  bring up the parabola, making the corrections to the continuum more and more severe.

Finally we want to analyze the slope of the parabola described by the correlators. In order to do that we can recast them into the following form given on the right hand side,

$$a^3 C(t) \equiv \int d^3x \sum_b \langle P^b(x, t) P^b(0) \rangle = A_P + B_P \left( \left( \left| \frac{t}{N_t} \right| - \frac{1}{2} \right)^2 - \frac{1}{24} \right) N_t a, \quad (2.35)$$

and study how the coefficient  $B_P$  depends on the LECs. As we can see clearly from figure 2. the main correction to the continuum limit comes from the modification of the constant  $A_P$  that determines the value of the minimum of the parabola. Indeed when the LEC  $c_2$  increases (we assume that at NLO the other LECs have the same effect to increase  $c_2^{\text{eff}}$ ), the minimum of the parabola becomes larger and larger compared to the continuum.

In figure 3. we look at the value of  $B_P$ , that is the coefficient that drives the slope of the parabola and thus is related to the masses of the pions. Differently from what happens in the GSM\* regime, in the Aoki regime also this parameter gets modified by lattice spacing effects. More in detail, plotting the value of  $B_P$  as a function of  $c_2$  it is interesting to note that for small enough values of  $c_2$  we are close to the continuum in the infinite-volume limit. Referring back to our discussion of the Aoki phase boundary in subsection 2.2, for  $-\infty < c_2 < c_2^*$  the system is in the so called Sharpe-Singleton scenario<sup>2</sup>. In that region flavor symmetry is not broken and all three pions remain massive. The situation changes when  $c_2$  is bigger than  $c_2^*$ , since the system enters in the Aoki phase. The value of  $B_P$  decreases quite rapidly to reach a lower limit of approximately 2/3 of the continuum limit. This is an indication that flavor symmetry is broken and as a consequence two of the three pions become massless.

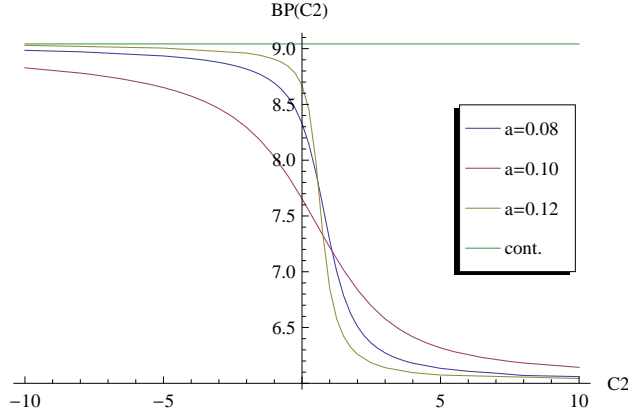
### 3 Staggered Chiral Perturbation Theory at NLO

#### 3.1 Introduction

In this section we study finite-volume corrections to the LECs in the framework of SChPT. As we will see, the situation is easier compared to WChPT since the corrections of order  $\mathcal{O}(am)$ ,  $\mathcal{O}(a^3)$  and  $\mathcal{O}(ap^2)$  don't appear in the staggered Lagrangian. This simplifies remarkably the computation and allows us to write the expression for the NLO partition function with a generic number of flavors in term of the LO one. Let us start to review briefly some basic known facts about SChPT and its equivalence to Staggered Chiral Random Matrix Theory (SChRMT) in the  $\epsilon$ -regime [22].

Leading Order $\mathcal{O}(\epsilon^0)$	$m, p^2, a^2$
Next-to-Leading Order $\mathcal{O}(\epsilon^2)$	—
Next-to-Next-to-Leading Order $\mathcal{O}(\epsilon^4)$	$m^2, mp^2, p^4, a^2m, a^2p^2, a^4$

<sup>2</sup>The value  $c_2^*$  is when  $c_2$  satisfies  $\hat{\epsilon}_V = 1$ , see the discussion before eq. (2.20).



**Figure 3.** The value of  $B_P$  from eq. (2.35) as function of  $c_2$ . Both  $x$ - and  $y$ -axis are in units of  $10^{10} \text{ MeV}^4$ .

**Table 3.** Contributions to the Staggered Chiral Lagrangian in the Aoki regime.

Staggered fermions are widely used to simulate quarks on the lattice. Indeed this formulation presents some clear advantages as the fact that the continuum chiral symmetry is not completely broken and that it is quite inexpensive to simulate numerically. However it doesn't solve completely the doubling problem. For every physical flavor there are four taste states that are degenerate in the continuum but split at finite lattice spacing because the taste symmetry is broken. Furthermore, in order to restore the correct number of degrees of freedom one has to introduce the so-called "rooting" procedure that consists of taking the fourth root of the fermion determinant.

The effective chiral Lagrangian that describes the staggered formulation including finite lattice size corrections has been introduced in [6] for the one flavor case and generalized to multiple flavors in [7]. The authors added all  $\mathcal{O}(a^2)$  terms to the continuum Lagrangian that are compatible with the staggered symmetries. As usual the breaking of the chiral symmetry from  $G = SU(4N_f)_R \times SU(4N_f)_L$  to  $SU(4N_f)_V$  is associated with the existence of light (pseudo) Goldstone boson fields that are collected into a  $4N_f \times 4N_f$  unitary matrix  $U$ . For example in the case  $N_f = 3$  the matrix  $U \in SU(12)$  can be parameterize as

$$U = \begin{pmatrix} u & \pi^+ & K^+ \\ \pi^- & d & K^0 \\ K^- & \bar{K}^0 & s \end{pmatrix}$$

where  $u, \pi^+, K + \dots$  are the  $4 \times 4$  matrices that take into account the taste degrees of freedom and that can be written in the Dirac basis as  $U = \sum_b U_b T_b$ , with denoting  $T_b = \{\xi_5, i\xi_\mu \xi_5, i\xi_\mu \xi_\nu, \xi_\mu, \xi_I\}$ . In the LCE regime the LO Lagrangian, which is of order  $\mathcal{O}(p^2, m, a^2)$ , reads as [7]

$$\begin{aligned}
\mathcal{L}_{\text{LO}} = & \frac{F^2}{8} \text{Tr} \left( \partial_\mu U \partial_\mu U^\dagger \right) - \frac{\Sigma}{4} \text{Tr} \left( M^\dagger U + U^\dagger M \right) - a^2 C_1 \text{Tr} \left( U \gamma_5 U^\dagger \gamma_5 \right) \\
& - a^2 \frac{C_3}{2} \sum_\mu [\text{Tr} (U \gamma_\mu U \gamma_\mu) + h.c.] - a^2 \frac{C_4}{2} \sum_\mu [\text{Tr} (U \gamma_{\mu 5} U \gamma_{\mu 5}) + h.c.] \\
& - a^2 \frac{C_{2V}}{4} \sum_\mu [\text{Tr} (U \gamma_\mu) \text{Tr} (U \gamma_\mu) + h.c.] - a^2 \frac{C_{2A}}{4} \sum_\mu [\text{Tr} (U \gamma_{\mu 5}) \text{Tr} (U \gamma_{\mu 5}) + h.c.] \\
& - a^2 \frac{C_{5V}}{4} \sum_\mu \left[ \text{Tr} (U \gamma_\mu) \text{Tr} (U^\dagger \gamma_\mu) \right] - a^2 \frac{C_{5A}}{4} \sum_\mu \left[ \text{Tr} (U \gamma_{\mu 5}) \text{Tr} (U^\dagger \gamma_{\mu 5}) \right], \\
& - a^2 C_6 \sum_{\mu < \nu} \text{Tr} \left( U \gamma_{\mu\nu} U^\dagger \gamma_{\mu\nu} \right), \tag{3.1}
\end{aligned}$$

where as usual  $F$  and  $\Sigma$  are the pion decay constant and the chiral condensate, respectively, while the  $4N_f \times 4N_f$  matrices  $\gamma_\mu$  are the generalizations of the ordinary  $4 \times 4$  Dirac matrices  $\xi_\mu$  (see [7] for details)<sup>3</sup>. In addition to the continuum Gasser-Leutwyler Lagrangian there are some taste-breaking contributions and as a consequence some new LECs usually denoted as  $C_1, C_{2A}, C_{2V}, C_3, C_4, C_{5A}, C_{5V}, C_6$ . For the one-flavor case the situation simplifies since all the two-trace terms in eq. (3.1) can be Fierz transformed into one-trace terms (see [6] for details). As we can see from table 3., if we want to go beyond LO, at NLO we will only get effects from one-loop finite-volume corrections of order  $\mathcal{O}(\epsilon^2)$ . Further LECs arising from the discretization effects of order  $\mathcal{O}(a^4)$ ,  $\mathcal{O}(a^2 p^2)$  and  $\mathcal{O}(a^2 m)$  will only appear at NNLO, being of order  $\mathcal{O}(\epsilon^4)$ . The corresponding terms are carefully listed in [55].

Quite recently it has been shown in [22] that in the  $\epsilon$ -regime SChPT is equivalent to SChRMT, including all one- and two-trace terms. In the latter theory the taste breaking terms are introduced by adding a taste diagonal matrix to the usual Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} \otimes \mathbb{I}_4 + a^2 \mathcal{T}, \tag{3.2}$$

with  $W$  a random matrix of size  $(N + \nu) \times N$  with complex entries, and where the explicit form of  $\mathcal{T}$  and its relations with all the terms in the taste breaking potential are given in table 1. of [22]. This correspondence is nothing else than the analogous relation for the staggered case between the  $\epsilon$ -regime of WChPT and WChRMT analyzed previously, valid at LO in the  $\epsilon$ -expansion. The aim of this section is to study the finite-volume correction to the LECs  $C_i$  following the same procedure utilized in the previous section 2 for WChPT. As a byproduct of the calculation we will find how the taste splittings, *i.e.* the difference between the mass square of the non-Goldstone boson and the Goldstone bosons, are modified.

### 3.2 Staggered Chiral Partition Function at $\mathcal{O}(\epsilon^2)$ for General $N_f$

In order to calculate the partition function at  $\mathcal{O}(\epsilon^2)$  in the staggered case we follow the same steps used in the analysis of the Wilson chiral Lagrangian. Let us start to rewriting

---

<sup>3</sup>There is also a mass term for the taste singlet pion that we have dropped.



the partition function as

$$\mathcal{Z} = \int_{SU(4N_f)} d_H U(x) e^{-S} = \int_{SU(4N_f)} D_H U_0 e^{-S_{U_0}} \mathcal{Z}_\xi(U_0), \quad (3.3)$$

where we have divided as usual the integration over the zero-modes  $U_0$  from the integration over the fluctuations  $\xi$  and where

$$\mathcal{Z}_\xi(U_0) = \int d\xi(x) \left( 1 - \frac{2}{3F^2V} \int d^4x \text{Tr} [\xi(x)^2] \right) e^{S_{U_0} - S}. \quad (3.4)$$

Expanding the function  $\mathcal{Z}_\xi(U_0)$  up to order  $\mathcal{O}(\epsilon^2)$  one can perform the Gaussian integrals over the fluctuations (for details about the expansion and the integration see appendix E). The next step is to reabsorb the finite-volume corrections into the LO Lagrangian by re-exponentiating all the terms found in the appendix E. At the end we can conclude that the NLO order partition function can be rewritten as the LO partition function with some renormalized LECs

$$\mathcal{Z}_{\text{NLO}} = \frac{\mathcal{N}'}{\mathcal{N}} \mathcal{Z}_{\text{LO}} \left( \Sigma^{\text{eff}}, C_i^{\text{eff}} \right). \quad (3.5)$$

These renormalized LECs following from the calculation in appendix E are shown in table 4. Similarly to the Wilson case the renormalized effective constants depend on the geometry of the system through the propagator  $\Delta(0)$ . As a further consequence we have extended the equivalence between SChPT and SChRMT up to NLO, as the form of the LO partition function is preserved for any number of flavors.

$\Sigma^{\text{eff}} = \Sigma \left( 1 - \frac{16N_f^2 - 1}{4F^2 N_f} \Delta(0) \right)$	$C_1^{\text{eff}} = C_1 \left( 1 - \frac{8N_f}{F^2} \Delta(0) \right)$
$C_{2V}^{\text{eff}} = C_{2V} - \frac{C_{2V}(16N_f^2 - 2) + 4C_3 N_f}{2N_f F^2} \Delta(0)$	$C_{2A}^{\text{eff}} = C_{2A} - \frac{C_{2A}(16N_f^2 - 2) + 4C_4 N_f}{2N_f F^2} \Delta(0)$
$C_3^{\text{eff}} = C_3 - \frac{C_3(16N_f^2 - 2) + 2C_{2V} N_f}{2N_f F^2} \Delta(0)$	$C_4^{\text{eff}} = C_4 - \frac{C_4(16N_f^2 - 2) + 2C_{2A} N_f}{2N_f F^2} \Delta(0)$
$C_{5V}^{\text{eff}} = C_{5V} \left( 1 - \frac{8N_f}{F^2} \Delta(0) \right)$	$C_{5A}^{\text{eff}} = C_{5A} \left( 1 - \frac{8N_f}{F^2} \Delta(0) \right)$
$C_6^{\text{eff}} = C_6 \left( 1 - \frac{8N_f}{F^2} \Delta(0) \right)$	

**Table 4.** The renormalized LECs in SChPT.

From the previous computation one can immediately understand how and if the finite-volume corrections affect the taste symmetry. Usually to study the taste symmetry violation one looks at the taste splitting  $\Delta$  in the pion sector, *i.e.* the difference between the mass square of a non-Goldstone pion and of the Goldstone one. At LO this quantity can

be derived from a tree level expansion of the chiral Lagrangian, and indeed the masses of the non-neutral meson<sup>4</sup> composed of quark  $b$  and  $c$  can be written as

$$m^2 = \mu(m_b + m_c) + a^2 \Delta_{\xi_B} . \quad (3.6)$$

The terms  $\Delta$  are related to the LECs through the relations [7]

$$\Delta_P = 0 , \quad (3.7)$$

$$\Delta_A = \frac{16}{F^2} (C_1 + 3C_3 + C_4 + 3C_6) , \quad (3.8)$$

$$\Delta_T = \frac{16}{F^2} (2C_3 + 2C_4 + 4C_6) , \quad (3.9)$$

$$\Delta_V = \frac{16}{F^2} (C_1 + C_3 + 3C_4 + 3C_6) , \quad (3.10)$$

$$\Delta_I = \frac{16}{F^2} (4C_3 + 4C_4) . \quad (3.11)$$

These splittings concern the Pseudoscalar ( $P$ ), Axial-Vector ( $A$ ), Tensor ( $T$ ), Vector ( $V$ ) and Singlet ( $I$ ) taste pions respectively. Since the LECs are modified at finite-volume the taste splitting get modified as follows

$$\Delta_P^{\text{NLO}} = \Delta_P = 0 , \quad (3.12)$$

$$\Delta_A^{\text{NLO}} = \Delta_A - \frac{16}{F^4} \left( 8N_f [C_1 + 3C_6] + \frac{[C_4 + 3C_3](16N_f^2 - 2) + 2[3C_{2V} + C_{2A}]N_f}{2N_f} \right) \Delta(0) , \quad (3.13)$$

$$\Delta_T^{\text{NLO}} = \Delta_T - \frac{16}{F^4} \left( 32N_f C_6 + \frac{[C_3 + C_4](16N_f^2 - 2) + 2[C_{2V} + C_{2A}]N_f}{N_f} \right) \Delta(0) , \quad (3.14)$$

$$\Delta_V^{\text{NLO}} = \Delta_V - \frac{16}{F^4} \left( 8N_f [C_1 + 3C_6] + \frac{[3C_4 + C_3](16N_f^2 - 2) + 2[C_{2V} + 3C_{2A}]N_f}{2N_f} \right) \Delta(0) , \quad (3.15)$$

$$\Delta_I^{\text{NLO}} = \Delta_I - \frac{32}{F^4} \left( \frac{[C_3 + C_4](16N_f^2 - 2) + 2[C_{2V} + C_{2A}]N_f}{N_f} \right) \Delta(0) . \quad (3.16)$$

---

<sup>4</sup>For flavor neutral mesons the situation is more complicated and other terms have to be introduced in the chiral lagrangian.

## 4 Summary and Discussion

In this paper we have computed  $\mathcal{O}(\epsilon^2)$  finite-volume corrections in the so-called  $\epsilon$ -regime as they arise in Wilson and Staggered Chiral Perturbation theory. Thus we have taken into account both corrections to infinite-volume and to the continuum limit. In SChPT  $\mathcal{O}(a^2)$  effects only enter at LO, parameterized through a large number of in total 6 new low-energy constants. In contrast in WChPT such  $\mathcal{O}(a^2)$  effects enter both at LO and at NLO in the  $\epsilon$ -expansion, leading to a total of 3 plus 9 LECs, respectively.

In consequence SChPT, although more complicated at LO, will remain simpler at NLO. In particular on the level of partition function the effect of NLO can be entirely expressed by renormalizing the LO LECs with corrections that we explicitly computed. As a second consequence the known equivalence between SChRMT and SChPT continues to hold at NLO. The drawback of the staggered formulation however remains, that the corresponding SChRMT has not been solved analytically to date. In addition our results for the NLO LECs provide us with the finite-volume corrections to the taste splittings as an application. Turning to the Wilson case much is known about the LO spectrum of the Wilson Dirac operator at fixed index, due to the equivalent WChRMT picture. This equivalence breaks down at NLO for an arbitrary number of flavors at fixed index including  $N_f = 2$ , as extra derivatives appear when trying to express the NLO partition function through the LO partition function with renormalized, effective couplings. Only for the original zero-mode group integral without fixing the index, and for the special case of  $SU(2)$ , the NLO partition functions keeps its functional form. As a consequence we can use the two corresponding effective couplings to quantify finite-volume and finite-lattice spacing effects on reaching the Aoki-phase. For the same reason the finite-volume and  $\mathcal{O}(a^2)$  corrections remain simplest for the two-point functions for  $SU(2)$ , which we have computed in the scalar and pseudoscalar case explicitly, and plotted in the pseudo-scalar sector for illustration.

Let us comment on finite-volume corrections to the positivity constraints on individual and certain combinations of LECs. At LO these were based on the positivity of the partition function at fixed index [11], on Hermiticity arguments for the generating functional for the spectral density of Wilson Dirac eigenvalues [25], and on the mass split using partially quenched WChPT [23]. It appears that neither line of argument can be easily translated to NLO, by simply replacing the LECs by effective ones. This has to do with the fact that at fixed index and/or for  $N_f > 2$  the functional form of the NLO partition function changes compared to LO.

In principle, the effect of NLO on the Wilson Dirac spectrum could be computed in the standard way, by introducing graded or replicated partition functions as generating functionals. However, due to the loss of determinantal structure of the partition function at NLO, that is observed at fixed index and certain vanishing couplings at LO (see appendix 2.4 where we also computed an extended version), and due to the loss of the WChPT-WChRMT relation, this seems to be a formidable task. Such a result would be very interesting in order to explain asymmetric effects on the spectrum attributed to NLO corrections in [17, 19]. On the other hand the extension of our computations to vector and axial-vector two-point correlation functions should be straightforward, both for  $SU(N_f)$  and for  $U(N_f)$  at fixed

index. The practical difficulty for the resulting expressions (except for  $SU(2)$ ) however is then to determine the large number of effective LECs from actual data, on which these quantities will depend.

**Acknowledgments:** We would like to thank Oliver Bär, Kim Splittorff and Edwin Laermann for fruitful discussions. Partial support by the SFB/TR12 “Symmetries and Universality in Mesoscopic Systems” of the German research council DFG is acknowledged (G.A.). F.P. thanks the G. Galilei Institute for Theoretical Physics in Florence for the hospitality. F.P. is supported by the Research Executive Agency (REA) of the European Union under Grant Agreement PITNGA- 2009-238353 (ITN STRONGnet).

## A Zero Mode Group Integral Identities

### A.1 General $SU(N_f)$ Case

In order to derive some  $SU(N_f)$  group identities among the expectation values of the various trace terms we follow the strategies adopted in [33, 51]. We introduce the differentiation with respect to the group elements  $U_{kl}$  of  $SU(N_f)$  defined as

$$\nabla_b \equiv i(t_b U)_{kl} \frac{\partial}{\partial U_{kl}} , \quad (\text{A.1})$$

where  $t_b$  are the generators of the algebra  $su(N_f)$  and satisfy the completeness relation

$$(t_b)_{ij}(t_b)_{kl} = \frac{1}{2} \left( \delta_{il}\delta_{jk} - \frac{1}{N_f} \delta_{ij}\delta_{kl} \right). \quad (\text{A.2})$$

This leads to the following derivatives

$$\nabla_b U = i t_b U , \quad \nabla_b U^\dagger = -i U^\dagger t_b . \quad (\text{A.3})$$

Considering that the Haar measure is left invariant, the integrals over total derivatives with respect to  $\nabla_b$  have to vanish and thus for example

$$0 = \int_{SU(N_f)} d_H U \nabla_c \left\{ \text{Tr}[t_c G(U)] \exp \left( \frac{m \Sigma V}{2} \text{Tr}[U + U^\dagger] - a^2 W_6 V \text{Tr}([U + U^\dagger])^2 - a^2 W_7 V \text{Tr}([U - U^\dagger])^2 - a^2 W_8 V \text{Tr}[U^2 + U^{\dagger 2}] \right) \right\} \quad (\text{A.4})$$

holds for any choice of the function  $G(U)$ . Throughout this appendix  $U = U_0$  is a constant matrix and for simplicity we drop the subscript compared to the main text. The following brackets denote the expectation value with respect to the integrand

$$\langle F(U) \rangle = \int_{SU(N_f)} d_H U F(U) e^{\frac{m \Sigma V}{2} \text{Tr}[U + U^\dagger] - a^2 W_6 V \text{Tr}([U + U^\dagger])^2 - a^2 W_7 V \text{Tr}([U - U^\dagger])^2 - a^2 W_8 V \text{Tr}[U^2 + U^{\dagger 2}]} . \quad (\text{A.5})$$

Choosing  $G(U) = U - U^\dagger$  we obtain the following identity

$$\begin{aligned}
0 = & \left( N_f - \frac{1}{N_f} \right) \langle \text{Tr} [U + U^\dagger] \rangle + \frac{m\Sigma V}{2} \left( \langle \text{Tr} [U^2 + U^{\dagger 2}] \rangle - 2N_f - \frac{1}{N_f} \langle (\text{Tr} [U - U^\dagger])^2 \rangle \right) \\
& - 2a^2 W_6 V \left\langle \text{Tr} [U + U^\dagger] \left( \text{Tr} [U^2 + U^{\dagger 2}] \rangle - 2N_f - \frac{1}{N_f} (\text{Tr} [U - U^\dagger])^2 \right) \right\rangle \\
& - 2a^2 W_7 V \left( \langle \text{Tr} [U^2 - U^{\dagger 2}] \text{Tr} [U - U^\dagger] \rangle - \frac{1}{N_f} \langle (\text{Tr} [U - U^\dagger])^2 \text{Tr} [U + U^\dagger] \rangle \right) \\
& - 2a^2 W_8 V \left( \langle \text{Tr} [U^3 + U^{\dagger 3}] \rangle - \langle \text{Tr} [U + U^\dagger] \rangle - \frac{1}{N_f} \langle \text{Tr} [U^2 - U^{\dagger 2}] \text{Tr} [U - U^\dagger] \rangle \right).
\end{aligned} \tag{A.6}$$

### A.2 The $SU(N_f = 2)$ Case

Here we rewrite the previous identity for the particular and more simple case of  $N_f = 2$  quarks. Indeed in this case some relations between the trace terms can be used to simplified considerably what we have found in (A.6). More in detail for any matrix  $U$  that belongs to the group  $SU(2)$  the following relations are valid

$$\begin{aligned}
\text{Tr}[U - U^\dagger] &= 0 & \text{Tr}[U^2 + U^{\dagger 2}] &= \frac{1}{2} \left( \text{Tr}[U + U^\dagger] \right)^2 - 4, \\
\text{Tr}[U^3 + U^{\dagger 3}] &= \frac{1}{4} \left( \text{Tr}[U + U^\dagger] \right)^3 - 3\text{Tr}[U + U^\dagger].
\end{aligned} \tag{A.7}$$

Note that these identities hold without taking an expectation value. Thus one can see that the general  $N_f$  expression of the identity reduces for this case to

$$\begin{aligned}
0 = & \left( \frac{3}{2} + 16a^2 c_2 V \right) \langle \text{Tr} [U + U^\dagger] \rangle + \frac{m\Sigma V}{4} \langle (\text{Tr} [U + U^\dagger])^2 \rangle - a^2 c_2 V \langle (\text{Tr} [U + U^\dagger])^3 \rangle \\
& - 4m\Sigma V
\end{aligned} \tag{A.8}$$

that is the analogon of eq. (A.6), and we recall that we denoted by  $c_2 = W_6 + W_8/2$ .

### A.3 General $U(N_f)$ Case

In this part of the appendix we derive  $U(N_f)$  group identities that will be useful when working at fixed topology. Differentiation with respect to the group elements  $U_{kl}$  of  $U \in U(N_f)$  is defined as before

$$\nabla_b \equiv i(t_b U)_{kl} \frac{\partial}{\partial U_{kl}}, \tag{A.9}$$

where  $t_b$  are now the generators of the algebra  $u(N_f)$  and satisfy the completeness relation

$$(t_b)_{ij}(t_b)_{kl} = \frac{1}{2} \delta_{il} \delta_{jk}. \tag{A.10}$$

Because of changing from  $SU(N_f)$  to  $U(N_f)$  group integrals we will have an extra factor  $\det[U]^\nu$  included in the integrand, the derivative of which reads

$$\nabla_b \det[U] = i \text{Tr}[t_b] \det[U]. \tag{A.11}$$

Again the integrals over total derivatives with respect to  $\nabla_b$  has to vanish:

$$0 = \int_{U(N_f)} d_H U \nabla_c \left\{ \text{Tr}[t_c G(U)] \det[U]^\nu \exp \left( \frac{m\Sigma V}{2} \text{Tr}[U + U^\dagger] + \frac{z\Sigma V}{2} \text{Tr}[U - U^\dagger] \right. \right. \\ \left. \left. - a^2 W_6 V \text{Tr} \left( [U + U^\dagger] \right)^2 - a^2 W_7 V \text{Tr} \left( [U - U^\dagger] \right)^2 - a^2 W_8 V \text{Tr} [U^2 + U^{\dagger 2}] \right) \right\} \quad (\text{A.12})$$

for any choice of  $G(U)$ . Here we have added an extra source term  $z$  for later convenience. The brackets denoting expectation values are now labeled by the index  $\nu$  in order to distinguish from the previous subsection. Also we use the following abbreviations:

$$m\Sigma V = \hat{m}, \quad z\Sigma V = \hat{z}, \quad \hat{a}_j^2 = a^2 W_j V \text{ for } j = 6, 7, 8.$$

We now derive a series of identities. Consider

- $t_c = t_0 \delta_{c,0}$  and  $G[U] = 1$ :

$$0 = \nu N_f + \frac{\hat{m}}{2} \langle \text{Tr}[U - U^\dagger] \rangle^\nu + \frac{\hat{z}}{2} \langle \text{Tr}[U + U^\dagger] \rangle^\nu \\ - 2(\hat{a}_6^2 + \hat{a}_7^2) \langle \text{Tr}[U + U^\dagger] \text{Tr}[U - U^\dagger] \rangle^\nu - 2\hat{a}_8^2 \langle \text{Tr}[U^2 - U^{\dagger 2}] \rangle^\nu, \quad (\text{A.13})$$

- $t_c = t_0 \delta_{c,0}$  and  $G[U] = U - U^\dagger$ :

$$0 = \langle \text{Tr}[U + U^\dagger] \rangle^\nu + \nu N_f \langle \text{Tr}[U - U^\dagger] \rangle^\nu + \frac{\hat{m}}{2} \langle (\text{Tr}[U - U^\dagger])^2 \rangle^\nu \\ + \frac{\hat{z}}{2} \langle \text{Tr}[U + U^\dagger] \text{Tr}[U - U^\dagger] \rangle^\nu - 2(\hat{a}_6^2 + \hat{a}_7^2) \langle \text{Tr}[U + U^\dagger] (\text{Tr}[U - U^\dagger])^2 \rangle^\nu \\ - 2\hat{a}_8^2 \langle \text{Tr}[U - U^\dagger] \text{Tr}[U^2 - U^{\dagger 2}] \rangle^\nu, \quad (\text{A.14})$$

- summing over  $t_c$  and  $G[U] = U - U^\dagger$ :

$$0 = N_f \langle \text{Tr}[U + U^\dagger] \rangle^\nu + \nu \langle \text{Tr}[U - U^\dagger] \rangle^\nu + \frac{\hat{m}}{2} \langle \text{Tr}[U^2 + U^{\dagger 2}] \rangle^\nu - \hat{m} N_f \\ + \frac{\hat{z}}{2} \langle \text{Tr}[U^2 - U^{\dagger 2}] \rangle^\nu - 2\hat{a}_6^2 \langle \text{Tr}[U^2 + U^{\dagger 2}] \text{Tr}[U + U^\dagger] \rangle^\nu + 4\hat{a}_6^2 N_f \langle \text{Tr}[U + U^\dagger] \rangle^\nu \\ - 2\hat{a}_7^2 \langle \text{Tr}[U^2 - U^{\dagger 2}] \text{Tr}[U - U^\dagger] \rangle^\nu - 2\hat{a}_8^2 \langle \text{Tr}[U^3 + U^{\dagger 3}] \rangle^\nu + 2\hat{a}_8^2 \langle \text{Tr}[U + U^\dagger] \rangle^\nu. \quad (\text{A.15})$$

We found two further identities with up to cubic powers of  $U$  and  $U^\dagger$  by choosing  $G[U] = U + U^\dagger$ . However, these identities contain new terms not present in the equations we wish to simplify and hence they are not useful.

## B Wilson Chiral Perturbation Theory for General $N_f$ at Fixed Index

In this section we focus on fixed index, the reason being that we then can compute the group integrals more explicitly. Also we have more group integral identities available from the previous subsection. In this way we can express the NLO partition function through

the LO one at renormalized couplings and derivatives thereof.

The partition function up to  $\mathcal{O}(\epsilon^2)$  can be written as a sum of two contribution  $S^{(0)}$  and  $S^{(2)}$  that read

$$\begin{aligned}
S^{(0)} &= +\frac{1}{2} \int d^4x \text{Tr} [\partial_\mu \xi \partial_\mu \xi] - \frac{1}{2} m \Sigma V \text{Tr} [U_0 + U_0^\dagger] - \frac{1}{2} z \Sigma V \text{Tr} [U_0 - U_0^\dagger] \\
&\quad + a^2 V W_8 \text{Tr} [U_0^2 + U_0^{\dagger 2}] + a^2 V W_6 \left( \text{Tr} [U_0 + U_0^\dagger] \right)^2 + a^2 V W_7 \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 \\
&\equiv S_{\partial^2}^{(0)} + S_{U_0}^{(0)} \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
S^{(2)} &= \frac{1}{12F^2} \int d^4x \text{Tr} [[\partial_\mu \xi, \xi][\partial_\mu \xi, \xi]] \\
&\quad + \frac{m \Sigma}{2F^2} \int d^4x \text{Tr} [(U_0 + U_0^\dagger) \xi^2] + \frac{z \Sigma}{2F^2} \int d^4x \text{Tr} [(U_0 - U_0^\dagger) \xi^2] \\
&\quad - 2a^2 \frac{W_8}{F^2} \int d^4x \text{Tr} [(U_0^2 + U_0^{\dagger 2}) \xi^2] - 2a^2 \frac{W_8}{F^2} \int d^4x \text{Tr} [U_0 \xi U_0 \xi + U_0^\dagger \xi U_0^\dagger \xi] \\
&\quad - 2a^2 \frac{W_6}{F^2} \int d^4x \left( \text{Tr} [(U_0 - U_0^\dagger) \xi] \right)^2 - 2a^2 \frac{W_6}{F^2} \int d^4x \text{Tr} [(U_0 + U_0^\dagger) \xi^2] \text{Tr} [U_0 + U_0^\dagger] \\
&\quad - 2a^2 \frac{W_7}{F^2} \int d^4x \left( \text{Tr} [(U_0 + U_0^\dagger) \xi] \right)^2 - 2a^2 \frac{W_7}{F^2} \int d^4x \text{Tr} [(U_0 - U_0^\dagger) \xi^2] \text{Tr} [U_0 - U_0^\dagger] \\
&\quad + 2a \frac{w_4}{F^2} \int d^4x \text{Tr} [\partial_\mu \xi \partial_\mu \xi] \text{Tr} [U_0 + U_0^\dagger] + 2a \frac{w_5}{F^2} \int d^4x \text{Tr} [\partial_\mu \xi \partial_\mu \xi (U_0 + U_0^\dagger)] \\
&\quad - a m w_6 V \left( \text{Tr} [U_0 + U_0^\dagger] \right)^2 - a m w_7 V \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 - a m w_8 V \text{Tr} [U_0^2 + U_0^{\dagger 2}] \\
&\quad + a^3 x_1 V \left( \text{Tr} [U_0 + U_0^\dagger] \right)^3 + a^3 x_2 V \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 \text{Tr} [U_0 + U_0^\dagger] + \\
&\quad + a^3 x_3 V \text{Tr} [U_0^2 + U_0^{\dagger 2}] \text{Tr} [U_0 + U_0^\dagger] + a^3 x_4 V \text{Tr} [U_0^2 - U_0^{\dagger 2}] \text{Tr} [U_0 - U_0^\dagger] \\
&\quad + a^3 x_5 V \text{Tr} [U_0^3 + U_0^{\dagger 3}] + a^3 x_6 V \text{Tr} [U_0 + U_0^\dagger]. \tag{B.2}
\end{aligned}$$

They are the contributions of order  $\mathcal{O}(\epsilon^0)$ , and  $\mathcal{O}(\epsilon^2)$  respectively, where we have split the former one into the zero-mode and propagating mode part. The order  $\mathcal{O}(\epsilon)$  vanishes.

Now we can proceed following the same steps used in the analysis of the two-flavor theory.

We begin rewriting the partition function for the  $N_f$  flavors with fixed index  $\nu$  as

$$Z^\nu = \int_{U(N_f)} [d_H U] \det[U_0]^\nu e^{-S} = \int_{U(N_f)} d_H U_0 \det[U_0]^\nu e^{-S_{U_0}^{(0)}} Z_\xi(U_0), \tag{B.3}$$

where

$$Z_\xi(U_0) = \int_{SU(N_f)} [d\xi(x)] \left( 1 - \frac{N_f}{3F^2 V} \int d^4x \text{Tr} [\xi(x)^2] \right) e^{S_{U_0}^{(0)} - S}. \tag{B.4}$$

The function  $Z_\xi(U_0)$  can be calculated expanding eq. (B.4) up to  $\mathcal{O}(\epsilon^2)$  order and then performing the integral over the Gaussian fluctuation  $\xi(x)$ . One obtains

$$\begin{aligned}
Z_\xi(U_0) = \mathcal{N} \Bigg\{ & 1 - \left( \frac{N_f^2 - 1}{N_f} \frac{mV\Sigma}{2F^2} \Delta(0) + a^3 x_6 V \right) \text{Tr} [U_0 + U_0^\dagger] - \frac{N_f^2 - 1}{N_f} \frac{zV\Sigma}{2F^2} \Delta(0) \text{Tr} [U_0 - U_0^\dagger] \\
& + \left( \frac{2a^2 V}{F^2} \left( W_8 \frac{N_f^2 - 2}{N_f} + W_6 + W_7 \right) \Delta(0) + amw_8 V \right) \text{Tr} [U_0^2 + U_0^{\dagger 2}] \\
& + \left( \frac{2a^2 V}{F^2} \left( W_6 \frac{N_f^2 - 1}{N_f} + \frac{N_f W_8 - 2W_7}{2N_f} \right) \Delta(0) + amw_6 V \right) \text{Tr} [U_0 + U_0^\dagger]^2 \\
& + \left[ \frac{2a^2 V}{F^2} \left( W_7 \frac{N_f^2 - 1}{N_f} + \frac{N_f W_8 - 2W_6}{2N_f} \right) \Delta(0) + amw_7 V \right] \text{Tr} [U_0 - U_0^\dagger]^2 \\
& - a^3 x_1 V \left( \text{Tr} [U_0 + U_0^\dagger] \right)^3 - a^3 x_2 V \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 \text{Tr} [U_0 + U_0^\dagger] \\
& - a^3 x_3 V \text{Tr} [U_0^2 + U_0^{\dagger 2}] \text{Tr} [U_0 + U_0^\dagger] - a^3 x_4 V \text{Tr} [U_0^2 - U_0^{\dagger 2}] \text{Tr} [U_0 - U_0^\dagger] \\
& - a^3 x_5 V \text{Tr} [U_0^3 + U_0^{\dagger 3}] \Bigg\}. \tag{B.5}
\end{aligned}$$

Using the relations (A.13)-(A.15) from the previous appendix A.3 we can rearrange several terms present in eq. (B.5) as a sum of other contributions. After several manipulation the final answer reads:

$$\begin{aligned}
Z_\xi(U_0) = \mathcal{N} \Bigg\{ & 1 - \left[ \frac{N_f^2 - 1}{N_f} \frac{mV\Sigma}{2F^2} \Delta(0) + \frac{a}{2W_8} \left( x_5 N_f + x_4 - x_5 \frac{W_7}{W_8} \right) + a^3 V \left( x_5 + x_6 + \frac{2x_5 N_f W_6}{W_8} \right) + \frac{x_5 z^2 \Sigma^2 V}{16aW_8^2} \right] \text{Tr} [U_0 + U_0^\dagger] \\
& - \left[ \frac{N_f^2 - 1}{N_f} \frac{zV\Sigma}{2F^2} \Delta(0) + \frac{a\nu}{2W_8} \left( x_5 + N_f \left( x_4 - x_5 \frac{W_7}{W_8} \right) \right) + \frac{x_5 m z \Sigma^2 V}{16aW_8^2} \right] \text{Tr} [U_0 - U_0^\dagger] \\
& + \left[ \frac{2a^2 V}{F^2} \left( W_8 \frac{N_f^2 - 2}{N_f} + W_6 + W_7 \right) \Delta(0) + amw_8 V - \frac{ax_5 m \Sigma V}{4W_8} \right] \text{Tr} [U_0^2 + U_0^{\dagger 2}] \\
& + \left[ \frac{2a^2 V}{F^2} \left( W_6 \frac{N_f^2 - 1}{N_f} + \frac{N_f W_8 - 2W_7}{2N_f} \right) \Delta(0) + amw_6 V \right] \left( \text{Tr} [U_0 + U_0^\dagger] \right)^2 \\
& + \left[ \frac{2a^2 V}{F^2} \left( W_7 \frac{N_f^2 - 1}{N_f} + \frac{N_f W_8 - 2W_6}{2N_f} \right) \Delta(0) + amV \left( w_7 - \left( x_4 - x_5 \frac{W_7}{W_8} \right) \frac{\Sigma}{4W_8} \right) \right] \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 \\
& - a^3 V \left( x_2 - \frac{W_6 + W_7}{W_8} \left( x_4 - x_5 \frac{W_7}{W_8} \right) \right) \left( \text{Tr} [U_0 - U_0^\dagger] \right)^2 \text{Tr} (U_0 + U_0^\dagger) \\
& - a^3 V \left( x_3 - \frac{x_5 W_6}{W_8} \right) \text{Tr} (U_0^2 + U_0^{\dagger 2}) \text{Tr} (U_0 + U_0^\dagger) - a^3 x_1 V \text{Tr} (U_0 + U_0^\dagger)^3 \\
& - \left( x_4 - x_5 \frac{(W_6 + 2W_7)}{W_8} \right) \frac{az\Sigma V}{4W_8} \text{Tr} (U_0 - U_0^\dagger) \text{Tr} (U_0 + U_0^\dagger) \Bigg\}. \tag{B.6}
\end{aligned}$$

At this point it turns out to be useful to define the new renormalized masses and LECs as

$$\begin{aligned}
\hat{m}^{\text{eff}} = \hat{m} - & \left[ \frac{(N_f^2 - 1)}{N_f} \frac{\hat{m}}{2F^2} \Delta(0) + \frac{\hat{a}}{2W_8 \sqrt{V}} \left( x_4 + x_5 \left( N_f - \frac{W_7}{W_8} \right) \right) \right. \\
& \left. + \frac{\hat{a}^3}{\sqrt{V}} \left( x_6 + x_5 \left( 1 + \frac{2N_f W_6}{W_8} \right) \right) + \frac{x_5 \hat{z}^2}{16\hat{a}W_8^2 \sqrt{V}} \right], \tag{B.7}
\end{aligned}$$

$$\hat{z}^{\text{eff}} = \hat{z} - \left[ \frac{(N_f^2 - 1)}{N_f} \frac{\hat{z}}{2F^2} \Delta(0) + \frac{\hat{a}\nu}{2W_8 \sqrt{V}} \left( x_5 + N_f \left( x_4 - x_5 \frac{W_7}{W_8} \right) \right) + \frac{x_5 \hat{m} \hat{z}}{16\hat{a}W_8^2 \sqrt{V}} \right] \tag{B.8}$$



$$(\hat{a}_8^{\text{eff}})^2 = \hat{a}_8^2 - \left[ \frac{2\hat{a}^2}{F^2} \left( W_8 \frac{N_f^2 - 2}{N_f} + W_6 + W_7 \right) \Delta(0) + \frac{\hat{a}\hat{m}}{\sqrt{V}} \left( \frac{w_8}{\Sigma} - \frac{x_5}{4W_8} \right) \right], \quad (\text{B.9})$$

$$(\hat{a}_6^{\text{eff}})^2 = \hat{a}_6^2 - \left[ \frac{2\hat{a}^2}{F^2} \left( W_6 \frac{(N_f^2 - 1)}{N_f} + \frac{N_f W_8 - 2W_7}{2N_f} \right) \Delta(0) + \frac{\hat{a}\hat{m}w_6}{\sqrt{V}} \right], \quad (\text{B.10})$$

$$\begin{aligned} (\hat{a}_7^{\text{eff}})^2 = \hat{a}_7^2 - & \left[ \frac{2\hat{a}^2}{F^2} \left( W_7 \frac{(N_f^2 - 1)}{N_f} + \frac{N_f W_8 - 2W_6}{2N_f} \right) \Delta(0) \right. \\ & \left. + \frac{\hat{a}\hat{m}}{\sqrt{V}} \left( \frac{w_7}{\Sigma} - \left( x_4 - x_5 \frac{W_7}{W_8} \right) \frac{1}{4W_8} \right) \right], \end{aligned} \quad (\text{B.11})$$

$$X_1^{\text{eff}} = x_1, \quad (\text{B.12})$$

$$X_2^{\text{eff}} = \left( x_2 - \frac{W_6 + W_7}{W_8} \left( x_4 - x_5 \frac{W_7}{W_8} \right) \right), \quad (\text{B.13})$$

$$X_3^{\text{eff}} = \left( x_3 - \frac{x_5 W_6}{W_8} \right), \quad (\text{B.14})$$

$$X_5^{\text{eff}} = \frac{\hat{a}\hat{z}}{4W_8\sqrt{V}} \left( x_4 - x_5 \frac{(W_6 + 2W_7)}{W_8} \right). \quad (\text{B.15})$$

Note that at NLO the quark and axial quark masses do not renormalize with the same LEC  $\Sigma$  any more. The renormalized constants  $\hat{m}^{\text{eff}}, \hat{m}_6^{\text{eff}}, \hat{a}_{6,7,8}^{\text{eff}}$  contain both  $\mathcal{O}(1)$  and  $\mathcal{O}(\epsilon^2)$  parts, the constants  $X_{1,2,3}^{\text{eff}}$  are all  $\mathcal{O}(1)$  only, and  $X_5^{\text{eff}}$  is of  $\mathcal{O}(\epsilon^2)$  only. This is because of the following NLO expression for the partition function:

$$\begin{aligned} \mathcal{Z}_{\text{NLO}}^\nu = & \frac{\mathcal{N}''}{\mathcal{N}} \left( \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}^{\text{eff}}, \hat{z}^{\text{eff}}, \hat{a}_6^{\text{eff}}, \hat{a}_7^{\text{eff}}, \hat{a}_8^{\text{eff}} \right) + X_1^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_6^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \right. \\ & + X_2^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_7^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) + X_3^{\text{eff}} \frac{2\hat{a}^3}{\sqrt{V}} \frac{\partial^2}{\partial \hat{a}_8^2 \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \\ & \left. - 4X_5^{\text{eff}} \frac{\partial^2}{\partial \hat{z} \partial \hat{m}} \mathcal{Z}_{\text{LO}}^\nu \left( \hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8 \right) \right). \end{aligned} \quad (\text{B.16})$$

## C Scalar and Pseudoscalar Currents in WChPT at Fixed Index

In this appendix we will complement the main body of this paper by computing the partition function and scalar and pseudoscalar two-point functions for an arbitrary number of flavors  $N_f$  at fixed index  $\nu$ . We begin with the partition function. Given the previous

appendix we only need to compute it to LO as the NLO one can be expressed through it. It is defined as

$$\mathcal{Z}_{\text{LO}}^{N_f, \nu}(\hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8) \equiv \int_{U(N_f)} d_H U \det[U]^\nu \exp \left[ \frac{\hat{m}}{2} \text{Tr} [U + U^\dagger] + \frac{\hat{z}}{2} \text{Tr} [U - U^\dagger] - \hat{a}_8^2 \text{Tr} [U^2 + U^{\dagger 2}] - \hat{a}_6^2 \left( \text{Tr} [U + U^\dagger] \right)^2 - \hat{a}_7^2 \left( \text{Tr} [U - U^\dagger] \right)^2 \right], \quad (\text{C.1})$$

with the rescaled quantities  $\hat{m} = mV\Sigma$ ,  $\hat{z} = zV\Sigma$ ,  $a^2 VW_j = \hat{a}_j^2$  for  $j = 6, 7, 8$ . We have dropped the index of  $U_0$  here and in the following. This integral has been calculated in the literature in a series of works [10, 11]. Let us briefly review and slightly extend their results.

Consider the following group integral that contains the above case for  $\hat{a}_6 = \hat{a}_7 = 0$ :

$$\mathcal{I}^{N_f, \nu} \equiv \int_{U(N_f)} d_H U \det[U]^\nu \exp \left[ \sum_{j=1}^{\infty} \left( \alpha_j \text{Tr}[U^j] + \beta_j \text{Tr}[U^{\dagger j}] \right) \right]. \quad (\text{C.2})$$

After diagonalizing the matrix,  $U \rightarrow v \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{N_f}}) v^\dagger$ , with  $v \in U(N_f)/U(1)^{N_f}$ , it can be written as a determinant over a single integral:

$$\begin{aligned} \mathcal{I}^{N_f, \nu} &= \mathcal{C}_{N_f} \int_{-\pi}^{\pi} \prod_{l=1}^{N_f} d\theta_l e^{i\theta_l} \exp \left[ \sum_{j=1}^{\infty} \left( \alpha_j e^{ij\theta_l} + \beta_j e^{-ij\theta_l} \right) \right] \prod_{k>n}^{N_f} |e^{i\theta_k} - e^{i\theta_n}|^2 \\ &= \mathcal{C}_{N_f} \int_{-\pi}^{\pi} \prod_{l=1}^{N_f} d\theta_l \det_{1 \leq n, k \leq N_f} \left[ \exp \left[ i(\nu + n - 1)\theta_k + \sum_{j=1}^{\infty} \alpha_j e^{ij\theta_k} \right] \right] \\ &\quad \times \det_{1 \leq n, k \leq N_f} \left[ \exp \left[ -i(n - 1)\theta_k + \sum_{j=1}^{\infty} \beta_j e^{ij\theta_k} \right] \right] \\ &= \mathcal{C}_{N_f} N_f! \det_{1 \leq n, k \leq N_f} \left[ \int_{-\pi}^{\pi} d\theta e^{i\theta(\nu + k - n)} e^{\sum_{j=1}^{\infty} (\alpha_j \exp[ij\theta] + \beta_j \exp[-ij\theta])} \right]. \quad (\text{C.3}) \end{aligned}$$

In the first step have rewritten the absolute value square of the Vandermonde determinant, the Jacobian resulting from the diagonalization, and pulled the exponential prefactors into the respective determinants. In the second step we have applied one of the de Bruijn integration formulas. The constant  $\mathcal{C}_{N_f}$  is the volume of the coset integral over  $v$ .

As a consequence for  $\hat{a}_6 = \hat{a}_7 = 0$  we can write the corresponding  $N_f$ -flavor partition function as a determinant of a single flavor partition function,

$$\mathcal{Z}_{\text{LO}}^{N_f, \nu}(\hat{m}, \hat{z}, 0, 0, \hat{a}_8) \sim \det \left[ \mathcal{Z}_{\text{LO}}^{N_f=1, \nu+k-n}(\hat{m}, \hat{z}, 0, 0, \hat{a}_8) \right]_{k, n=1, \dots, N_f}, \quad (\text{C.4})$$

where

$$\mathcal{Z}_{\text{LO}}^{N_f=1, \nu}(\hat{m}, \hat{z}, 0, 0, \hat{a}_8) = \mathcal{C}_1 \int_{-\pi}^{\pi} \exp[i\theta\nu + \hat{m} \cos(\theta) + i\hat{z} \sin(\theta) - 2\hat{a}_8^2 \cos(2\theta)] d\theta. \quad (\text{C.5})$$

In the particular case of two degenerate flavors eq. (C.4) reduces to

$$\mathcal{Z}^{N_f=2,\nu} \sim (\mathcal{Z}^{N_f=1,\nu})^2 - \mathcal{Z}^{N_f=1,\nu+1} \mathcal{Z}^{N_f=1,\nu-1}, \quad (\text{C.6})$$

where we have suppressed the arguments. The remaining two LECs  $\hat{a}_{6,7}$  can be switched on by performing two Gaussian integrals on the above formulas, following [11]. In addition it has been shown in [25] based on Hermiticity that both LECs have to be non-positive,  $\hat{a}_6, \hat{a}_7 \leq 0$ . Consequently we obtain the following expression:

$$\mathcal{Z}_{\text{LO}}^{N_f,\nu}(\hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8) = \int_{-\infty}^{\infty} \frac{dy_6 dy_7}{16\pi |\hat{a}_6 \hat{a}_7|} e^{-\frac{y_6^2}{16\hat{a}_6^2}} e^{-\frac{y_7^2}{16\hat{a}_7^2}} \mathcal{Z}_{\text{LO}}^{N_f,\nu}(\hat{m} - y_6, \hat{z} - y_7, 0, 0, \hat{a}_8). \quad (\text{C.7})$$

Now let's derive the partition function at fixed topology in the case of two flavors. In fact the Gaussian integration over the partition function (2.22) can be performed explicitly, and we obtain that

$$\begin{aligned} \mathcal{Z}_{\text{LO}}^{2,\nu}(\hat{m}, \hat{z}, \hat{a}_6, \hat{a}_7, \hat{a}_8) &= \frac{\mathcal{C}_2}{2} \int_{-\pi}^{\pi} d\theta_1 d\theta_2 e^{i(\theta_1+\theta_2)\nu} \left(1 - e^{i(\theta_1-\theta_2)}\right) e^{\hat{m}(\cos\theta_1+\cos\theta_2)+i\hat{z}(\sin\theta_1+\sin\theta_2)} \\ &\times e^{4\hat{a}_6^2(\cos\theta_1+\cos\theta_2)^2} e^{-4\hat{a}_7^2(\sin\theta_1+\sin\theta_2)^2} e^{-2\hat{a}_8^2(\cos 2\theta_1+\cos 2\theta_2)}. \end{aligned} \quad (\text{C.8})$$

Although in this expression the integrals do not factorize it is very useful for a numerical integration. The normalization constant  $\mathcal{C}_2$  is not important as it drops out in expectation values.

We can now compute the NLO two-point functions for general  $N_f$  at fixed index, following the same lines as in the previous appendix B. In fact written in terms of group averages  $\langle \dots \rangle^\nu$ , where the superscript denotes the index, the expressions with or without fixing the index don't differ. This is because the propagating modes that we contract always live in  $SU(N_f)$ . The only difference is that for fixed index at  $N_f = 2$  we no longer have the  $SU(2)$  identities at hand, e.g.  $\text{Tr}[U - U^\dagger] \neq 0$  no longer applies.

For simplicity we will only present that unflavored scalar and flavored pseudoscalar two-point functions as in the main text. The results we obtain are

$$\begin{aligned} \langle S_0(x) S_0(0) \rangle^\nu &= \frac{(\Sigma^{\text{eff}})^2}{4} \left\langle \left( \text{Tr}[U + U^\dagger] \right)^2 \right\rangle_{\text{NLO}}^\nu - \frac{\hat{a} \Sigma c_3}{\sqrt{V}} \left\langle \text{Tr}[U + U^\dagger]^3 \right\rangle_{\text{LO}}^\nu \\ &- \frac{\Sigma^2}{2F^2} \left\{ \left\langle \text{Tr}[U^2 + U^{\dagger 2}] \right\rangle_{\text{LO}}^\nu - 2N_f - \frac{1}{N_f} \left\langle \left( \text{Tr}[U - U^\dagger] \right)^2 \right\rangle_{\text{LO}}^\nu \right\} \Delta(x). \end{aligned} \quad (\text{C.9})$$

For the flavored pseudoscalars we sum over all generators  $t_b$  of  $su(N_f)$

$$\begin{aligned} \sum_b \langle P_b(x) P_b(0) \rangle^\nu &= -\frac{(\Sigma^{\text{eff}})^2}{8} \left\{ \left\langle \text{Tr}[U^2 + U^{\dagger 2}] \right\rangle_{\text{NLO}}^\nu - 2N_f - \frac{1}{N_f} \left\langle \left( \text{Tr}[U - U^\dagger] \right)^2 \right\rangle_{\text{NLO}}^\nu \right\} \\ &+ \frac{\hat{a} c_3 \Sigma}{2\sqrt{V}} \left\langle \left( \text{Tr}[U^2 + U^{\dagger 2}] - \frac{1}{N_f} \left( \text{Tr}[U - U^\dagger] \right)^2 - 2N_f \right) \text{Tr}[U + U^\dagger] \right\rangle_{\text{LO}}^\nu \\ &+ \frac{\Sigma^2}{4F^2} \left\{ \frac{1}{2} \left\langle \left( \text{Tr}[U - U^\dagger] \right)^2 \right\rangle_{\text{LO}}^\nu + \frac{(N_f^2 + 2)}{2N_f^2} \left\langle \left( \text{Tr}[U + U^\dagger] \right)^2 \right\rangle_{\text{LO}}^\nu \right. \\ &\quad \left. + 2N_f^2 - \frac{2}{N_f} \left\langle \text{Tr}[U^2 + U^{\dagger 2}] \right\rangle_{\text{LO}}^\nu \right\} \Delta(x). \end{aligned} \quad (\text{C.10})$$

Note that in deriving the expression for the zero-momentum correlation functions, which are functions only of the Euclidean time  $t$ , we would have to make use of the relations

$$\int d^3x \Delta(x-y) = \frac{aN_T}{2} \left[ \left( \left| \frac{t_0}{T} \right| - \frac{1}{2} \right)^2 - \frac{1}{24} \right]. \quad (\text{C.11})$$

## D Explicit Computation of Partition Function and Currents for $SU(2)$

In this appendix we will derive explicit integral representations of the scalar and pseudoscalar current densities whose formal expressions are given in subsection 2.4. Since we are dealing with the two-flavor case we can describe the group manifold using the familiar parameterization of  $SU(2)$

$$U_0 = (\cos \alpha + i \hat{n} \cdot \sigma \sin \alpha) \quad (\text{D.1})$$

where  $\hat{n}$  is a three-dimensional unit vector, the  $\sigma$ 's are the Pauli matrices and  $0 < \alpha < 2\pi$ . With this parameterization for an arbitrary element of  $SU(2)$ , the normalized group measure is

$$\int dU_0 = \frac{1}{2\pi^2} \int d\Omega_{\hat{n}} \int_0^{2\pi} d\alpha \sin(\alpha)^2. \quad (\text{D.2})$$

The partition function can thus be written in a more manageable way using this parameterization as

$$\mathcal{Z}_{\text{NLO}} = \frac{\mathcal{C}'}{2\pi^2} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \exp \left[ 2m\Sigma^{\text{eff}} V \cos(\alpha) - 16a^2 c_2^{\text{eff}} V \cos(\alpha)^2 \right], \quad (\text{D.3})$$

and correspondingly for LO by dropping the superscript eff and having a different normalization constant  $\mathcal{C}$ . As a check we obtain for  $c_2 = 0$  the known result for the equal mass  $SU(2)$  partition function  $\mathcal{Z}_{\text{LO}} = \mathcal{C} I_1(2\hat{m})/(2\hat{m}\pi)$  in terms of a modified Bessel function. The expressions for the two-point scalar and pseudoscalar current correlators derived in subsection 2.4 can be rewritten as

$$\begin{aligned} \langle S_0(x) S_0(0) \rangle &= \frac{(\Sigma^{\text{eff}})^2 \mathcal{C}'}{\mathcal{Z}_{\text{NLO}} 2\pi^2} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \cos(\alpha)^2 e^{2\hat{m}^{\text{eff}} \cos(\alpha) - 16\hat{a}^2 c_2^{\text{eff}} \cos(\alpha)^2} \\ &\quad - \frac{4\Sigma^2 \mathcal{C} \Delta(x)}{F^2 2\pi^2 \mathcal{Z}_{\text{LO}}} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \left( \cos(\alpha)^2 - 1 \right) e^{2\hat{m} \cos(\alpha) - 16\hat{a}^2 c_2 \cos(\alpha)^2} \\ &\quad - \frac{64\hat{a} \Sigma c_3}{\sqrt{V}} \frac{\mathcal{C}}{2\pi^2 \mathcal{Z}_{\text{LO}}} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \cos(\alpha)^3 e^{2\hat{m} \cos(\alpha) - 16\hat{a}^2 c_2 \cos(\alpha)^2}, \quad (\text{D.4}) \end{aligned}$$

$$\begin{aligned} \langle P_b(x) P_b(0) \rangle &= -\frac{(\Sigma^{\text{eff}})^2 \mathcal{C}'}{2\pi^2 \mathcal{Z}_{\text{NLO}}} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \left( \cos(\alpha)^2 - 1 \right) e^{2\hat{m}^{\text{eff}} \cos(\alpha) - 16\hat{a}^2 c_2^{\text{eff}} \cos(\alpha)^2} \\ &\quad + \frac{\Sigma^2 \mathcal{C} \Delta(x)}{F^2 2\pi^2 \mathcal{Z}_{\text{LO}}} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \left( \cos(\alpha)^2 + 2 \right) e^{2\hat{m} \cos(\alpha) - 16\hat{a}^2 c_2 \cos(\alpha)^2} \\ &\quad + \frac{16\hat{a} c_3 \Sigma \mathcal{C}}{\sqrt{V} \pi^2 \mathcal{Z}_{\text{LO}}} \int_0^{2\pi} d\alpha \sin(\alpha)^2 \left( \cos(\alpha)^3 - \cos(\alpha) \right) e^{2\hat{m} \cos(\alpha) - 16\hat{a}^2 c_2 \cos(\alpha)^2}. \quad (\text{D.5}) \end{aligned}$$

## E Staggered Chiral Perturbation Theory for General $N_f$

In the following we report explicitly all the terms arising in the  $\epsilon$ -expansion up to order  $\mathcal{O}(\epsilon^2)$  of the partition function  $\mathcal{Z}_\xi(U_0)$  defined in the SChPT subsection 3.2. At LO  $\mathcal{O}(\epsilon^0)$  one obtains

$$\begin{aligned}
S^{(0)} = & +\frac{1}{4} \int d^4x \text{Tr} [\partial_\mu \xi \partial_\mu \xi] - \frac{\Sigma V}{4} \text{Tr} [M^\dagger U_0 + U_0^\dagger M] - a^2 V C_1 \text{Tr} (\gamma_5 U_0 \gamma_5 U_0^\dagger) \\
& - a^2 \frac{V C_3}{2} \sum_\mu [\text{Tr} (U_0 \gamma_\mu U_0 \gamma_\mu) + h.c.] - a^2 \frac{V C_4}{2} \sum_\mu [\text{Tr} (U_0 \gamma_{\mu 5} U_0 \gamma_{\mu 5}) + h.c.] \\
& - a^2 \frac{C_{2V}}{4} \sum_\mu [\text{Tr} (U_0 \gamma_\mu) \text{Tr} (U_0 \gamma_\mu) + h.c.] - a^2 \frac{V C_{2A}}{4} \sum_\mu [\text{Tr} (U_0 \gamma_{\mu 5}) \text{Tr} (U_0 \gamma_{\mu 5}) + h.c.] \\
& - a^2 \frac{V C_{5V}}{4} \sum_\mu [\text{Tr} (U_0 \gamma_\mu) \text{Tr} (U_0^\dagger \gamma_\mu)] - a^2 \frac{V C_{5A}}{4} \sum_\mu [\text{Tr} (U_0 \gamma_{\mu 5}) \text{Tr} (U_0^\dagger \gamma_{\mu 5})] \\
& - a^2 V C_6 \sum_{\mu < \nu} \text{Tr} [U_0 \gamma_{\mu\nu} U_0^\dagger \gamma_{\mu\nu}], \tag{E.1}
\end{aligned}$$

while for the first order  $\mathcal{O}(\epsilon)$  we have  $S^{(1)} = 0$ , and for the second order one finds

$$\begin{aligned}
S^{(2)} = & \frac{1}{24F^2} \int d^4x \text{Tr} [[\partial_\mu \xi, \xi][\partial_\mu \xi, \xi]] + \frac{\Sigma}{4F^2} \int d^4x \text{Tr} [M^\dagger U_0 \xi^2 + \xi^2 U_0^\dagger M] \\
& + \int d^4x \left[ -2a^2 \frac{C_1}{F^2} \text{Tr} (\gamma_5 U_0 \xi \gamma_5 \xi U_0^\dagger) + a^2 \frac{C_1}{F^2} \text{Tr} (\gamma_5 U_0 \xi^2 \gamma_5 U_0^\dagger) + a^2 \frac{C_1}{F^2} \text{Tr} (\gamma_5 U_0 \gamma_5 \xi^2 U_0^\dagger) \right. \\
& + a^2 \frac{C_3}{F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_\mu U_0 \xi \gamma_\mu) + h.c.] + a^2 \frac{C_3}{F^2} \sum_\mu [\text{Tr} (U_0 \xi^2 \gamma_\mu U_0 \gamma_\mu) + h.c.] \\
& + a^2 \frac{C_4}{F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_{\mu 5} U_0 \xi \gamma_{\mu 5}) + h.c.] + a^2 \frac{C_4}{F^2} \sum_\mu [\text{Tr} (U_0 \xi^2 \gamma_{\mu 5} U_0 \gamma_{\mu 5}) + h.c.] \\
& + a^2 \frac{C_{2V}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_\mu) \text{Tr} (U_0 \xi \gamma_\mu) + h.c.] + a^2 \frac{C_{2V}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi^2 \gamma_\mu) \text{Tr} (U_0 \gamma_\mu) + h.c.] \\
& + a^2 \frac{C_{2A}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_{\mu 5}) \text{Tr} (U_0 \xi \gamma_{\mu 5}) + h.c.] + a^2 \frac{C_{2A}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi^2 \gamma_{\mu 5}) \text{Tr} (U_0 \gamma_{\mu 5}) + h.c.] \\
& - a^2 \frac{C_{5V}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_\mu) \text{Tr} (\xi U_0^\dagger \gamma_\mu) - \frac{1}{2} \text{Tr} (U_0 \xi^2 \gamma_\mu) \text{Tr} (U_0^\dagger \gamma_\mu) - \frac{1}{2} \text{Tr} (U_0 \gamma_\mu) \text{Tr} (\xi^2 U_0^\dagger \gamma_\mu)] \\
& - a^2 \frac{C_{5A}}{2F^2} \sum_\mu [\text{Tr} (U_0 \xi \gamma_{\mu 5}) \text{Tr} (\xi U_0^\dagger \gamma_{\mu 5}) - \frac{1}{2} \text{Tr} (U_0 \xi^2 \gamma_{\mu 5}) \text{Tr} (U_0^\dagger \gamma_{\mu 5}) - \frac{1}{2} \text{Tr} (U_0 \gamma_{\mu 5}) \text{Tr} (\xi^2 U_0^\dagger \gamma_{\mu 5})] \\
& \left. - 2a^2 \frac{C_6}{F^2} \sum_{\mu < \nu} \text{Tr} (U_0 \xi \gamma_{\mu\nu} \xi U_0^\dagger \gamma_{\mu\nu}) + a^2 \frac{C_6}{F^2} \sum_{\mu < \nu} [\text{Tr} (U_0 \xi^2 \gamma_{\mu\nu} U_0^\dagger \gamma_{\mu\nu}) + \text{Tr} (U_0 \gamma_{\mu\nu} \xi^2 U_0^\dagger \gamma_{\mu\nu})] \right].
\end{aligned}$$

Now one can perform the Gaussian integrals over the fluctuations, and one finds that

$$\begin{aligned}
Z_{\xi(U_0)} = & \mathcal{N} \left( 1 - \frac{mV\Sigma}{4F^2} \frac{16N_f^2-1}{4N_f} \Delta(0) \text{Tr} [U_0 + U_0^\dagger] - \frac{8a^2 C_1 V}{F^2} N_f \Delta(0) \text{Tr} [U_0 \gamma_5 U_0^\dagger \gamma_5] \right. \\
& - \frac{a^2 [C_3(16N_f^2-2)+2C_{2V}N_f]}{4N_f F^2} \Delta(0) V \sum_\mu [\text{Tr} (U_0 \gamma_\mu U_0 \gamma_\mu) + h.c.] \\
& - \frac{a^2 [C_4(16N_f^2-2)+2C_{2A}N_f]}{4N_f F^2} \Delta(0) V \sum_\mu [\text{Tr} (U_0 \gamma_{\mu 5} U_0 \gamma_{\mu 5}) + h.c.] \\
& \left. - \frac{a^2 [C_{2V}(16N_f^2-2)+8C_3N_f]}{8N_f F^2} \Delta(0) V \sum_\mu [\text{Tr} (U_0 \gamma_\mu) \text{Tr} (U_0 \gamma_\mu) + h.c.] \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{a^2[C_{2A}(16N_f^2-2)+8C_4N_f]}{8N_f F^2} \Delta(0)V \sum_{\mu} [\text{Tr}(U_0\gamma_{\mu 5}) \text{Tr}(U_0\gamma_{5\mu}) + h.c.] \\
& -\frac{4a^2C_{5V}N_f}{2F^2} \Delta(0)V \sum_{\mu} \text{Tr}(U_0\gamma_{\mu}) \text{Tr}(U_0^{\dagger}\gamma_{\mu}) \\
& -\frac{4a^2C_{5A}N_f}{2F^2} \Delta(0)V \sum_{\mu} \text{Tr}(U_0\gamma_{\mu 5}) \text{Tr}(U_0^{\dagger}\gamma_{5\mu}) \\
& -\frac{8a^2C_6N_f}{F^2} \Delta(0)V \sum_{\mu<\nu} \text{Tr}[U_0\gamma_{\mu\nu}U_0^{\dagger}\gamma_{\mu\nu}].
\end{aligned}$$

Because in this appendix we didn't have to use any group integral identities the same relations hold for fixed index, by adding  $\det[U_0^{\nu}]$  inside the zero-mode group integral.

## References

- [1] S. R. Sharpe and R. L. Singleton, Jr, Phys. Rev. D **58** (1998) 074501 [hep-lat/9804028].
- [2] O. Bär, G. Rupak and N. Shoresh, Phys. Rev. D **70** (2004) 034508 [arXiv:hep-lat/0306021].
- [3] G. Rupak and N. Shoresh, Phys. Rev. D **66** (2002) 054503 [hep-lat/0201019].
- [4] S. Aoki and O. Bär, Phys. Rev. D **70** (2004) 116011 [arXiv:hep-lat/0409006].
- [5] S. Aoki, Phys. Rev. D **68** (2003) 054508 [arXiv:hep-lat/0306027].
- [6] W. J. Lee and S. R. Sharpe, Phys. Rev. D **60** (1999) 114503 [arXiv:hep-lat/9905023].
- [7] C. Aubin and C. Bernard, Phys. Rev. D **68** (2003) 034014 [arXiv:hep-lat/0304014].
- [8] J. Gasser and H. Leutwyler, Phys. Lett. B **184** (1987) 83.
- [9] S. R. Sharpe, Phys. Rev. D **74** (2006) 014512 [arXiv:hep-lat/0606002].
- [10] P. H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. Lett. **105** (2010) 162002 [arXiv:1001.2937 [hep-th]].
- [11] G. Akemann, P. H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **83** (2011) 085014 [arXiv:1012.0752 [hep-lat]].
- [12] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **84** (2011) 065031 [arXiv:1105.6229 [hep-lat]].
- [13] G. Akemann and T. Nagao, JHEP **1110** (2011) 060 [arXiv:1108.3035 [math-ph]].
- [14] M. Kieburg, J. J. M. Verbaarschot and S. Zafeiropoulos, Phys. Rev. Lett. **108** (2012) 022001 [arXiv:1109.0656 [hep-lat]].
- [15] G. Akemann and A. C. Ipsen, JHEP **1204** (2012) 102 [arXiv:1202.1241 [hep-lat]].
- [16] M. Kieburg, J. Phys. A **45** (2012) 205203 [arXiv:1202.1768 [math-ph]].
- [17] P. H. Damgaard, U. M. Heller and K. Splittorff, Phys. Rev. D **85** (2012) 014505 [arXiv:1110.2851 [hep-lat]].
- [18] A. Deuzeman, U. Wenger and J. Wuilloud, JHEP **1112** (2011) 109 [arXiv:1110.4002 [hep-lat]].
- [19] P. H. Damgaard, U. M. Heller and K. Splittorff, arXiv:1206.4786 [hep-lat].
- [20] P. H. Damgaard, J. C. Osborn, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. B **547** (1999) 305 [hep-th/9811212].
- [21] F. Basile and G. Akemann, JHEP **0712** (2007) 043 [arXiv:0710.0376 [hep-th]].

- [22] J. C. Osborn, Phys. Rev. D **83** (2011) 034505 [arXiv:1012.4837 [hep-lat]].
- [23] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **85** (2012) 014503 [arXiv:1111.2404 [hep-lat]].
- [24] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **85** (2012) 054504 [arXiv:1112.3998 [hep-lat]].
- [25] M. Kieburg, K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **85** (2012) 094011 [arXiv:1202.0620 [hep-lat]].
- [26] S. Aoki and O. Bär, PoS LAT **2007** (2007) 062 [arXiv:0710.0072 [hep-lat]].
- [27] S. R. Sharpe and J. M. S. Wu, Phys. Rev. D **70** (2004) 094029 [hep-lat/0407025].
- [28] S. R. Sharpe and J. M. S. Wu, Phys. Rev. D **71** (2005) 074501 [hep-lat/0411021].
- [29] O. Bär, S. Necco and S. Schaefer, JHEP **0903** (2009) 006 [arXiv:0812.2403 [hep-lat]].
- [30] A. Shindler, Phys. Lett. B **672** (2009) 82 [arXiv:0812.2251 [hep-lat]].
- [31] S. Necco and A. Shindler, JHEP **1104** (2011) 031 [arXiv:1101.1778 [hep-lat]].
- [32] P. H. Damgaard, T. DeGrand and H. Fukaya, JHEP **0712** (2007) 060 [arXiv:0711.0167 [hep-lat]].
- [33] G. Akemann, F. Basile and L. Lellouch, JHEP **0812**, 069 (2008) [arXiv:0804.3809 [hep-lat]].
- [34] C. Lehner and T. Wettig, JHEP **0911** (2009) 005 [arXiv:0909.1489 [hep-lat]].
- [35] C. Lehner, S. Hashimoto and T. Wettig, JHEP **1006** (2010) 028 [arXiv:1004.5584 [hep-lat]].
- [36] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.
- [37] S. Weinberg, Physica A **96** (1979) 327.
- [38] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.
- [39] S. Aoki, O. Bär and B. Biedermann, Phys. Rev. D **78** (2008) 114501 [arXiv:0806.4863 [hep-lat]].
- [40] F. Bernardoni, J. Bulava and R. Sommer, PoS LATTICE **2011** (2011) 095 [arXiv:1111.4351 [hep-lat]].
- [41] S. Aoki and A. Gocksch, Phys. Rev. D **45** (1992) 3845.
- [42] S. Aoki and A. Gocksch, Phys. Lett. B **231** (1989) 449.
- [43] S. Aoki and A. Gocksch, Phys. Lett. B **243** (1990) 409.
- [44] K. Jansen *et al.* [XLF Collaboration], Phys. Lett. B **624** (2005) 334 [hep-lat/0507032].
- [45] S. Aoki, A. Ukawa and T. Umemura, Phys. Rev. Lett. **76** (1996) 873 [hep-lat/9508008].
- [46] L. Del Debbio, L. Giusti, M. Lüscher, R. Petronzio and N. Tantalo, JHEP **0702** (2007) 056 [hep-lat/0610059].
- [47] L. Del Debbio, L. Giusti, M. Lüscher, R. Petronzio and N. Tantalo, JHEP **0702** (2007) 082 [hep-lat/0701009].
- [48] F. Farchioni, R. Frezzotti, K. Jansen, I. Montvay, G. C. Rossi, E. Scholz, A. Shindler and N. Ukita *et al.*, Eur. Phys. J. C **39** (2005) 421 [hep-lat/0406039].
- [49] F. Farchioni, K. Jansen, I. Montvay, E. Scholz, L. Scorzato, A. Shindler, N. Ukita and C. Urbach *et al.*, Eur. Phys. J. C **42** (2005) 73 [hep-lat/0410031].
- [50] F. Farchioni, K. Jansen, I. Montvay, E. E. Scholz, L. Scorzato, A. Shindler, N. Ukita and

- C. Urbach *et al.*, Phys. Lett. B **624** (2005) 324 [hep-lat/0506025].
- [51] P. H. Damgaard, M. C. Diamantini, P. Hernandez and K. Jansen, Nucl. Phys. B **629** (2002) 445 [arXiv:hep-lat/0112016].
- [52] P. H. Damgaard, P. Hernández, K. Jansen, M. Laine and L. Lellouch, Nucl. Phys. B **656** (2003) 226 [arXiv:hep-lat/0211020].
- [53] F. C. Hansen, Nucl. Phys. B **345** (1990) 685.
- [54] S. Aoki, O. Bär and S. R. Sharpe, Phys. Rev. D **80** (2009) 014506 [arXiv:0905.0804 [hep-lat]].
- [55] S. R. Sharpe and R. S. Van de Water, Phys. Rev. D **71** (2005) 114505 [hep-lat/0409018].